Norwegian University of Science and Technology

Department of Mathematical Sciences

# Examination paper for
# TMA4305 Partial Differential Equations

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**Examination date:** 19 December 2016

**Examination time (from–to):** 09:00 – 13:00

**Permitted examination support material:** One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he wants. No other aids permitted.

**Other information:**
There are 7 problems of equal weight: 1, 2, 3, 4, 5, 6a, 6b.

**Language:** English

**Number of pages:** 2

**Number of pages enclosed:** 0

**Checked by:**

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| **Informasjon om trykking av eksamensoppgave** |
| **Originalen er:** |
| **1-sidig** ☐ **2-sidig** ☐ |
| **sort/hvit** ☐ **farger** ☐ |
| **skal ha flervalgskjema** ☐ |

Date Signature

**Problem 1** Solve the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + tx^2 \\ v(x,0) = 0 \end{cases}$$

where $-\infty < x < \infty, \ t > 0$.

**Problem 2** Consider the equation

$$u_t = u_{xx} + 2u_{yy}$$

in the domain $\Omega_T \equiv \Omega \times (0,T)$, where $\Omega$ is a bounded domain in the $xy-$plane. Formulate and prove the *parabolic* maximum principle for the solutions. (You may assume[1] that $u \in C^2(\overline{\Omega_T})$.)

**Problem 3** Show that the problem

$$\begin{cases} u_{tt} - c^2 \Delta u + 3u = F(\mathbf{x},t), \\ u(\mathbf{x},0) = f(\mathbf{x}) \\ u_t(\mathbf{x},0) = g(\mathbf{x}) \end{cases}$$

can have at most one smooth solution $u = u(\mathbf{x},t)$ in $\mathbb{R}^3 \times (0,T)$. Assume that for each fixed $t \geq 0$ there is a radius $R_t$ such that $u(\mathbf{x},t) = 0$ when $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq R_t$. *Hint:* Energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left( w_t^2 + c^2 |\nabla w|^2 + 3w^2 \right) dx_1 dx_2 dx_3.$$

**Problem 4** Determine the constants $C$ and $\alpha$ so that $u(x) = Ce^{\alpha|x|}$ is a solution of the equation

$$u''(x) - k^2 u(x) = -\delta(x)$$

in the sense of distributions. Use test functions in $C_0^\infty(\mathbb{R})$.

[1] The notation means that $u$ has continuous second derivatives up to the boundary

**Problem 5** The function

$$u(x,t) = \begin{cases} \dfrac{x-2}{t+2}, & \text{when} \quad x > \xi(t) \\ 0, \quad \text{when} & x < \xi(t) \end{cases}$$

is a weak solution to Burgers's equation

$$u_t + uu_x = 0$$

in the upper half-plane $t > 0$. The unknown shock curve $x = \xi(t)$ starts at the origin. Determine the shock curve and draw a picture with characteristics in the $xt$-plane.

**Problem 6** Assume that the variational integral

$$I(u) = \int_0^1 \int_0^1 \left(e^{2x} u_x^2 + u_x u_y + u_y^2 - 2u^3\right) dx\, dy$$

attains its minimum among all smooth functions with boundary values 0 on the sides of the square.

**a)** ~~Prove that the minimizer is unique.~~ Due to a misprint, this question is impossible to do.

**b)** Derive a second order equation for the minimizer (the Euler–Lagrange Equation). Then, prove that the minimizer is non-negative.

Good luck!