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PARTIAL DIFFERENTIAL EQUATIONS

TMA4305 Exam. 1. XII. 2014

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① The Cauchy–Riemann equations imply

$$\Delta u = 0, \quad \Delta v = 0, \quad \nabla u \cdot \nabla v = 0.$$

Now

$$\nabla(uv) = u\nabla v + v \nabla u,$$

$$\Delta(uv) = \nabla\cdot(u\nabla v + v\nabla u) = u\Delta v + \nabla u \cdot \nabla v + v\Delta u = 0+0+0 = 0.$$

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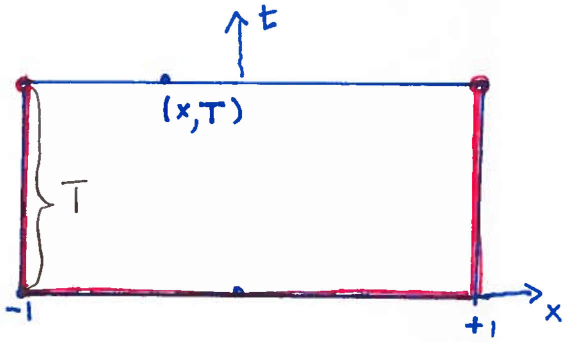
② 1) "At an interior maximum point



$$u_t = 0, \quad u_x = 0, \quad u_{xx} \le 0.$$

This contradicts the equation

$$0 = u_t = u_{xx} + u_x^2 - 1 \le 0 + 0 - 1 = -1.$$

Thus there is no interior maximum.

2) If (x,T), $-1 < x < 1$, is a maximum point we have

$$u_t \ge 0, \quad u_x = 0, \quad u_{xx} \le 0.$$

Again the contradiction $0 \le -1$ arises.

Thus the maximum is attained on the parabolic boundary.

(3a) $u_{tt} = c^2\Delta u - m^2 u$ (Klein-Gordon)

$$E(t) = \frac{1}{2}\iiint_{\mathbb{R}^3} \left(u_t^2 + c^2|\nabla u|^2 + m^2u^2 \right) dx\,dy\,dz$$

$$\frac{dE}{dt} = \iiint_{\mathbb{R}^3} \Big(u_t \underbrace{u_{tt}}_{c^2\Delta u - m^2 u} + c^2 \nabla u \cdot \nabla u_t + m^2 u u_t \Big) dx\,dy\,dz$$

$$= \iiint_{\mathbb{R}^3} \left(u_t c^2 \Delta u + c^2 \underline{\nabla u \cdot \nabla u_t} \right) dx\,dy\,dz$$

Integrate over a large ball B_{R^*}:

$$\underbrace{\iint_{\partial B_{R^*}} u_t \frac{\partial u}{\partial \nu}\, dS}_{=0 \text{ when } R^* > R.} = \iiint_{B_{R^*}} \left[u_t \Delta u + \underline{\nabla u_t \cdot \nabla u} \right] dx\,dy\,dz \quad [\text{Green I}]$$

It follows that

$$\frac{dE}{dt} = \iiint 0\, dx\,dy\,dz = 0$$

and so $\underline{E(t) \equiv \text{Constant}}$.

(3b) If there are two solutions u_1 and u_2, then $u = u_2 - u_1$ satisfies

$$\begin{cases} u_{tt} = c^2\Delta u - m^2 u \\ u_t = 0 \text{ at } t = 0 \\ u = 0 \text{ at } t = 0 \implies \nabla u(x,y,z,0) = \bar{0}. \end{cases}$$

$$E(0) = \frac{1}{2}\iiint_{\mathbb{R}^3} \left(0^2 + c^2|\bar{0}|^2 + m^2 0^2 \right) dx\,dy\,dz = 0$$

From $E(t) = E(0) = 0$ (by 3a) we conclude that

$$u_t \equiv 0, \quad \nabla u \equiv 0.$$

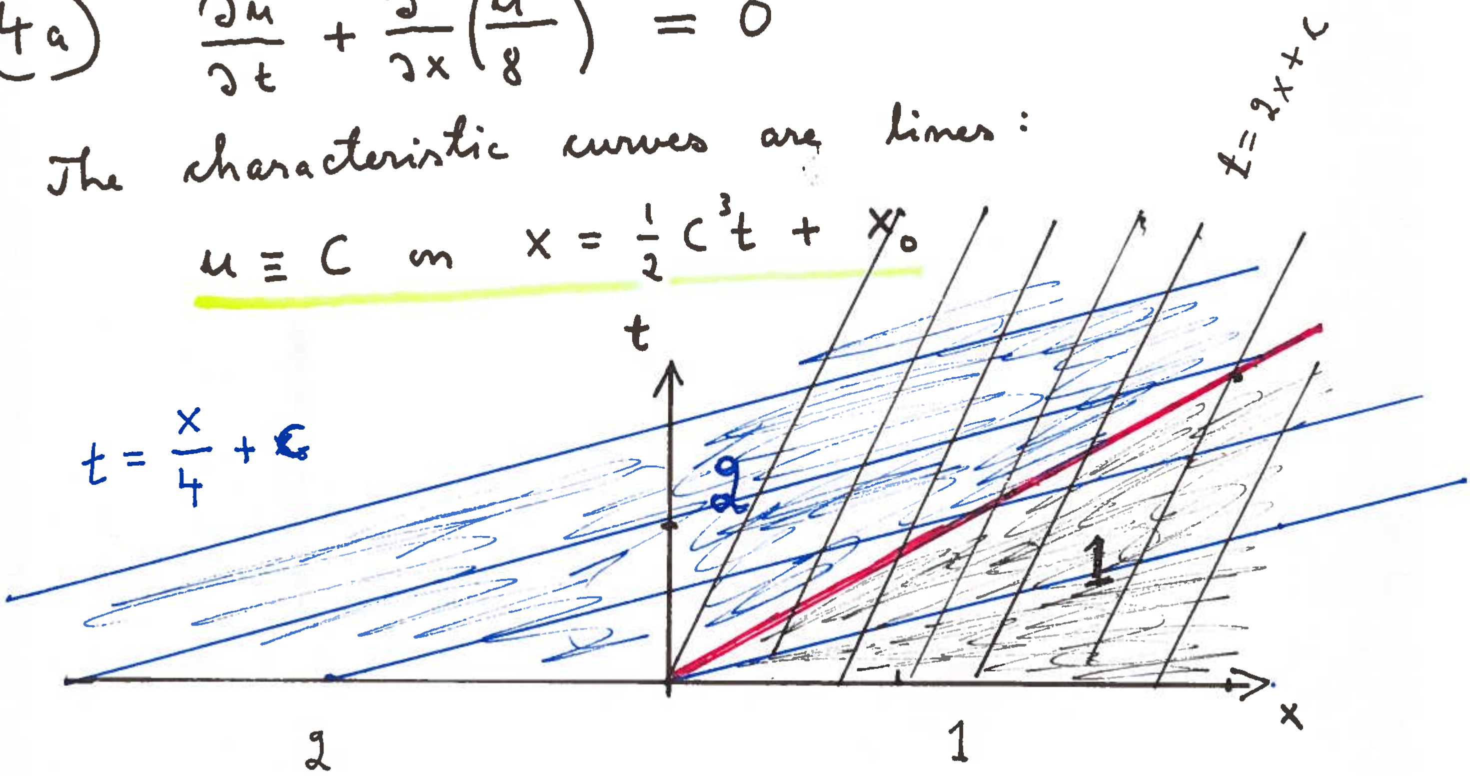
It follows that $u \equiv$ Constant. Since $u = 0$ at time $t = 0$, we have $u \equiv 0$. Thus

$$u_2 = u_1. \quad \text{QED}$$

4a $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^4}{8}\right) = 0$$

The characteristic curves are lines:

$$u \equiv C \text{ on } x = \frac{1}{2}C^3 t + x_0$$

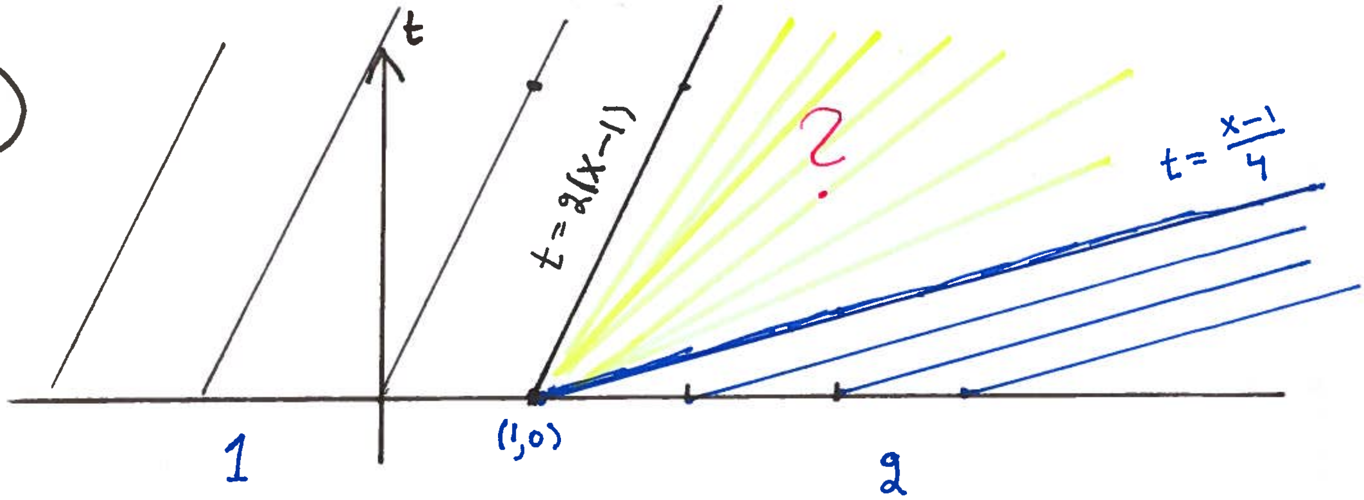


The Rankine-Hugoniot shock condition yields

$$\frac{dx}{dt} = \frac{1^4/8 - 2^4/8}{1-2} = \frac{15}{8}$$

$$\boxed{t = \frac{8}{15}x}$$ The shock curve

4b



$$u(x,t) = \psi\left(\frac{x-1}{t}\right) \quad \text{rarefation wave}$$

Insert into the equation:

$$2\left(-\frac{x-1}{\cancel{t^2}}\right)\psi' + \psi^3 \cdot \frac{1}{\cancel{t}}\psi' = 0$$

Discard the case $\psi' = 0$, which does not fit.

$$-2\tau + \psi^3(\tau) = 0, \quad \tau = \frac{x-1}{t}$$

It follows that

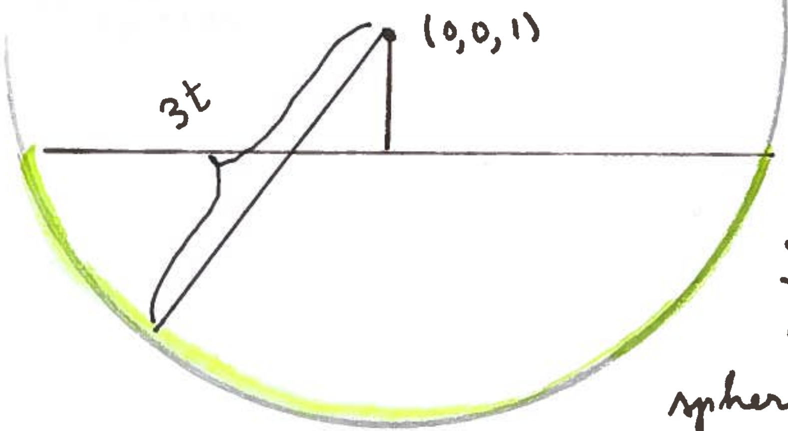
$$\boxed{u(x,t) = \sqrt[3]{2\,\frac{x-1}{t}}}$$

in the sector $2(x-1) > t > \frac{1}{4}(x-1)$, $x > 1$.

<u>Solution</u>

$$u(x,t) = \begin{cases} 1, & t > 2(x-1), \quad t > 0 \\ \sqrt[3]{2\,\frac{x-1}{t}}, & 2(x-1) > t > \frac{1}{4}(x-1) \\ 2, & 0 < t < \frac{x-1}{4} \end{cases}$$

⑤ $$\frac{\partial^2 v}{\partial t^2} = 3^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$



By Kirchhoff's formula only the initial data on a sphere of radius $ct = 3t$ and center at $(0,0,1)$ count. Thus $v(0,0,1,t) = 0$ when $0 < t \le 1/3$. We get

$$\underline{v(0,0,1,t) \neq 0 \text{ when } t > \tfrac{1}{3}}.$$

KIRCHHOFF:

$$v(0,0,1,t) = t \cdot \frac{1}{4\pi \cdot 9t^2} \iint\limits_{\substack{z<0 \\ x^2+y^2+(z-1)^2 = 3t}} \left(1 - e^{-x^2-y^2}\right) dS_{3t}$$

approaches the average over a half-sphere × t

We cut out a small cap near the South Pole

$$\geq t \cdot \frac{1}{4\pi\, 9t^2} \iint\limits_{\substack{z<0 \\ x^2+y^2+(z-1)^2 = 3t \\ x^2+y^2 \geq 1}} (1 - e^{-1})\, dS_{3t}$$

Approximatively the average of $1-e^{-1}$ over a half-sphere.

$$\approx t \cdot \frac{1}{2}(1-e^{-1}) \longrightarrow \infty \quad \text{as } t \to \infty.$$

$$\boxed{\lim_{t\to\infty} v(0,0,1,t) = \infty} \qquad (!)$$

(The initial condition sets 50% of the Universe in motion.)