NTNU

Department of Mathematical Sciences

Examination paper for
TMA4305 Partial Differential Equations

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Examination date: December 9, 2013
Examination time (from–to): 09:00–13:00
Permitted examination support material: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Language: English
Number of pages: 2
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 Consider the first order differential equation

$$\begin{aligned} u_x + uu_y &= -ku, \quad x > 0,\ y \in \mathbb{R}, \\ u(0,y) &= u_0(y), \quad y \in \mathbb{R}. \end{aligned}$$

Here $k > 0$ is a given constant and $u_0 \in C^1(\mathbb{R})$ is a given function.

a) Compute the characteristics for this equation.

b) Give a simple condition on u_0 in order that the equation always has a local solution.

c) Determine the solution in the case $u_0(y) = y$.

Problem 2 Consider the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= \cos(x+t), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = x, \quad u_t(x,0) &= \sin(x), \quad x \in \mathbb{R}. \end{aligned}$$

Compute the solution.

Problem 3 Consider the initial-boundary value problem

$$\begin{aligned} u_t &= \Delta u, \quad x \in B, \quad t > 0, \\ u(x,0) &= 0, \quad x \in B, \\ u(\sigma,t) &= 1, \quad |\sigma| = 1, \end{aligned} \tag{1}$$

where $B = \{x \in \mathbb{R}^3 \mid |x| < 1\}$.

a) Show that $v = u - 1$ satisfies

$$\begin{aligned} v_t &= \Delta v, \quad x \in B, \quad t > 0, \\ v|_{t=0} &= -1, \\ v(\sigma,t) &= 0, \quad |\sigma| = 1. \end{aligned}$$

b) Assume that v is a radially symmetric function, $v = v(r)$, and show that v satisfies

$$(rv)_t = (rv)_{rr}.$$

You may use the fact that the Laplacian in three dimensions reads in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(\frac{1}{\sin^2(\psi)}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\psi^2} + \cot(\psi)\frac{\partial}{\partial\psi}\Big)$$

c) Introduce the function $w = rv$, and show that it satisfies

$$\begin{aligned} w_t &= w_{rr}, \quad r \in (0,1), \quad t > 0, \\ w|_{t=0} &= -r, \\ w(0,t) &= w(1,t) = 0, \quad t > 0, \end{aligned}$$

and solve this system by separation of variables.

d) Determine the formal solution of (1).

Problem 4 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$.

a) Show that

$$-\Delta(\pm u + \frac{|x|^2}{2n}\lambda) \leq 0,$$

where $\lambda = \max_{\overline{\Omega}} |f|$.

b) Show that

$$\max_{\overline{\Omega}} |u| \leq C\Big(\max_{\overline{\Omega}} |f| + \max_{\partial\Omega} |g|\Big)$$

where the constant C only depends on Ω.
You may use the following result: Assume that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $-\Delta w \leq 0$ in Ω. Then $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$.