Solutions
for
TMA4305 Partial Differential Equations

December 17, 2012.

Problem 1

a) We find

$$w_t = \frac{2CD}{L^2}, \quad w_x = \frac{2C}{L^2}x, \quad w_{xx} = \frac{2C}{L^2}$$

which implies that $w_t - Dw_{xx} = 0$.

b) Consider the function $v = u - w$, which satisfies, in addition to required smoothness assumptions,

$$v_t - Dv_{xx} \leq 0.$$

The maximum principle (Salsa, Theorem 2.2, p. 32) then states that

$$\max_{R_L} v = \max_{\partial_p R_L} v.$$

On the parabolic boundary we find

$$v(x,0) = u(x,0) - M_0 - \frac{2C}{L^2}x^2 \leq 0,$$
$$v(\pm L, t) = u(\pm L, t) - M_0 - C - \frac{2CD}{L^2}t \leq 0$$

since all constants are assumed positive. Hence we conclude that

$$u(x,t) - w(x,t) = v(x,t) \leq 0, \quad (x,t) \in R_L.$$

c) Choose $L \geq |x_0|$. We have

$$u(x_0, t_0) \leq w(x_0, t_0) = \frac{2C}{L^2}\big(\frac{x_0^2}{2} + Dt_0\big) + M_0.$$

If we let $L \to \infty$, we conclude that

$$u(x_0, t_0) \leq M_0.$$

d) *Theorem.* Given a function u that is continuous on $\mathbb{R} \times [0, T]$ such that the derivatives u_x, u_{xx}, u_t are continuous on $\mathbb{R} \times (0, T)$. Assume that

$$u_t - Du_{xx} \leq 0$$

on $\mathbb{R} \times (0, T)$, and

$$(1) \qquad u(x,t) \leq C, \quad (x,t) \in \mathbb{R} \times [0,T].$$

Here C and D are all positive constants. Then

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} u(x,t) = \sup_{x\in\mathbb{R}} u(x,0),$$

assuming that $\sup_{x\in\mathbb{R}} u(x,0) > 0$.

The assumption $\sup_{x\in\mathbb{R}} u(x,0) > 0$ is not necessary. Let $M_0 = \sup_{\mathbb{R}} u(x,0)$ as before. If M_0 is negative, replace C by $C_0 = 2\max\{C, |M_0|\}$ everywhere. Then the parabolic estimate goes as follows

$$\begin{aligned} v(\pm L, t) &= u(\pm L, t) - M_0 - C_0 - \frac{2C_0 D}{L^2} t \\ &\leq (u(\pm L, t) - \frac{1}{2}C_0) - (M_0 + \frac{1}{2}C_0) \leq 0, \end{aligned}$$

and the rest of the argument is as before.

Problem 2

a) (See Salsa, Theorem 5.1, p. 264.) Let u and v be two solutions. Define $w = u - v$, which then satisfies

$$w_{tt} - c^2 \Delta w = 0 \text{ in } \Omega_T, \quad w = 0 \text{ on } \Omega.$$

The boundary conditions are

$$w = 0 \text{ on } \partial_D \Omega \times [0,T), \quad \partial_\nu w = 0 \text{ on } \partial_N \Omega \times [0,T).$$

Define the energy

$$E(t) = \frac{1}{2} \int_\Omega \left(w_t^2 + c^2 |\nabla w|^2 \right) dx$$

with derivative

$$\dot{E}(t) = \int_\Omega \left(w_{tt} - c^2 \Delta w \right) w_t \, dx + c^2 \int_{\partial\Omega} (\partial_\nu w) w_t dS.$$

Given the assumptions on w, we see that $\dot{E}(t) = 0$ for all $t \in [0,T]$. Since $E(0) = 0$, we conclude that $E(t) = 0$ for all $t \in [0,T]$. Thus w is identically zero.

Problem 3

a) The hardest computation is given in the problem. It only remains to write

$$\Delta_x^2 v(x,\xi) = -\Delta_x \left(\frac{1}{2\pi}(1 + \ln r) \right) = \Delta_x \Phi(x - \xi) = -\delta(x - \xi)$$

since $\Phi(x) = -\ln|x|/(2\pi)$ is the fundamental solution for the Laplacian (Salsa, p. 125).

b) Green's identity (Salsa, eqn. (3.57), p. 132 or eqn. (1.15), p. 12) states that

$$\int_{\partial\Omega} \Big(v\partial_\nu u - u\partial_{\nu_x} v \Big)\, dS_x = \int_\Omega \Big(v\Delta u - u\Delta_x v \Big)\, dS_x.$$

It only remains to replace u by Δu. If you worry that v is not in $C^2(\bar{\Omega})$, you first integrate over $\Omega \setminus \{y \mid |y-\xi| \le \epsilon\}$, and let $\epsilon \to 0$.

c) Start with

$$\begin{aligned}
u(\xi) &= -\int_\Omega \Phi(x-\xi)\Delta u(x)\,dx - \int_{\partial\Omega} \Big(u\partial_{\nu_x}\Phi(x-\xi) - \Phi(x-\xi)\partial_\nu u \Big)\, dS_x \\
&= -\int_\Omega (\Delta_x v(x,\xi) + \frac{1}{2\pi})\Delta u(x)\,dx \\
&\quad - \int_{\partial\Omega} \Big(u\partial_{\nu_x}(\Delta_x v(x,\xi) + \frac{1}{2\pi}) - (\Delta_x v(x,\xi) + \frac{1}{2\pi})\partial_\nu u \Big)\, dS_x \\
&= -\int_\Omega \Delta_x v(x,\xi)\Delta u(x)\,dx \\
&\quad - \int_{\partial\Omega} \Big(u\partial_{\nu_x}\Delta_x v(x,\xi) - \Delta_x v(x,\xi)\partial_\nu u \Big)\, dS_x \\
&\quad - \frac{1}{2\pi}\Big(\int_\Omega \Delta u\,dx - \int_{\partial\Omega} \partial_\nu u\,dS \Big) \\
&= -\int_\Omega \Delta_x v(x,\xi)\Delta u(x)\,dx \\
&\quad - \int_{\partial\Omega} \Big(u\partial_{\nu_x}\Delta_x v(x,\xi) - \Delta_x v(x,\xi)\partial_\nu u \Big)\, dS_x.
\end{aligned}$$

Here we have used Green's identity to conclude that $\int_\Omega \Delta u\,dx - \int_{\partial\Omega} \partial_\nu u\,dS = 0$. If we replace $\int_\Omega \Delta_x v\Delta u\,dx$ using **b)** above, we find

$$\begin{aligned}
u(\xi) &= -\int_\Omega \Delta_x v(x,\xi)\Delta u(x)\,dx - \int_{\partial\Omega} \Big(u(x)\partial_{\nu_x}\Delta_x v(x,\xi) - \Delta_x v(x,\xi)\partial_\nu u(x) \Big)\, dS_x \\
&= -\int_\Omega v(x,\xi)\Delta^2 u(x)\,dx \\
&\quad + \int_{\partial\Omega} \Big(v(x,\xi)\partial_\nu \Delta u(x) - \Delta u\partial_{\nu_x} v(x,\xi) \\
&\qquad\qquad - u(x)\partial_{\nu_x}\Delta_x v(x,\xi) + \Delta_x v(x,\xi)\partial_\nu u(x) \Big)\, dS_x,
\end{aligned}$$

which is what we wanted to prove.

d) First we use **b)** above with v replaced by φ_ξ. Then we use **b)** above with u replaced by φ_ξ and v replaced by u. Thus

$$\begin{aligned}\int_\Omega \varphi_\xi \Delta^2 u\,dx &= \int_\Omega \Delta_x\varphi_\xi \Delta u\,dx \\ &\quad + \int_{\partial\Omega} \big(\varphi_\xi \partial_\nu \Delta u - \Delta u \partial_{\nu_x}\varphi_\xi\big)\,dS_x \\ &= \int_\Omega \Delta_x^2 \varphi_\xi u(x)\,dx \\ &\quad + \int_{\partial\Omega} \big(\Delta_x\varphi_\xi \partial_\nu u - u\partial_{\nu_x}\Delta_x\varphi_\xi\big)\,dS_x \\ &\quad + \int_{\partial\Omega} \big(\varphi_\xi \partial_\nu \Delta u - \Delta u \partial_{\nu_x}\varphi_\xi\big)\,dS_x \\ &= \int_{\partial\Omega} \big(\Delta_x\varphi_\xi \partial_\nu u - u\partial_{\nu_x}\Delta_x\varphi_\xi\big)\,dS_x \\ &\quad + \int_{\partial\Omega} \big(\varphi_\xi \partial_\nu \Delta u - \Delta u \partial_{\nu_x}\varphi_\xi\big)\,dS_x,\end{aligned}$$

where we finally used that $\Delta_x^2\varphi_\xi = 0$.

Subtracting this equality from the previous one, and using the properties of u we find

$$\begin{aligned}u(\xi) = &-\int_\Omega G(x,\xi)f(x)\,dx \\ &+ \int_{\partial\Omega} \Big(G(x,\xi)\partial_\nu \Delta u(x) - \Delta u(x)\partial_{\nu_x} G(x,\xi)\Big)\,dS_x,\end{aligned}$$

Using the properties of G we conclude that the result holds.