Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations

Exam, December 17, 2012, Time: 9:00–13:00.

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Grades will be announced January 17, 2013.

Aids: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Problem 1

a) Define

$$w(x,t) = \frac{2C}{L^2}\Big(\frac{x^2}{2} + Dt\Big) + M_0.$$

Here D, C, L, and M_0 are constants. Show that

$$w_t - Dw_{xx} = 0.$$

b) Given a function u that is continuous on $\mathbb{R} \times [0,T]$ such that the derivatives u_x, u_{xx}, u_t are continuous on $\mathbb{R} \times (0,T)$. Assume that

$$\begin{aligned} u_t - Du_{xx} &\leq 0 \text{ on } \mathbb{R} \times (0,T),\ D > 0, \\ u &\leq C \text{ on } \mathbb{R} \times [0,T],\ C > 0. \end{aligned} \tag{1}$$

Let $M_0 = \sup_{\mathbb{R}} u(x,0) > 0$. Use the maximum principle to show that $w \geq u$ on $R_L = [-L,L] \times [0,T]$.

c) Let (x_0,t_0) be an arbitrary point in $\mathbb{R} \times [0,T]$, and choose L such that $(x_0,t_0) \in R_L$. Show why we can conclude that

$$u(x_0,t_0) \leq M_0.$$

d) State the theorem that you have proved now. Is the assumption $M_0 > 0$ necessary?

Problem 2 Let

$$Q_T = \Omega \times (0, T)$$

where $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary. Assume that u satisfies

$$u_{tt} - c^2 \Delta u = f \text{ in } Q_T, \tag{2}$$

and

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad x \in \Omega. \tag{3}$$

Boundary conditions are

$$u = \phi_1 \text{ on } \partial_D \Omega \times [0, T), \quad \partial_\nu u = \phi_2 \text{ on } \partial_N \Omega \times [0, T). \tag{4}$$

Here $\partial_N \Omega$ is a relatively open subset of $\partial\Omega$, and $\partial_D \Omega = \partial\Omega \setminus \partial_N \Omega$, and f, g, h, ϕ_1, ϕ_2 are given functions of sufficient regularity.

a) Show that there is at most one solution u in $C^{2,2}(Q_T) \cap C^{1,1}(\bar{Q}_T)$ of the above problem (2)–(4).

Problem 3 Define the function

$$v(x, \xi) = -\frac{r^2}{8\pi} \ln r, \quad x, \xi \in \mathbb{R}^2, x \neq \xi, r = |x - \xi|.$$

a) Show that v is fundamental solution of the biharmonic equation, that is,

$$\Delta_x^2 v(x, \xi) = \Delta_x(\Delta_x v(x, \xi)) = -\delta(x - \xi)$$

in $\mathbb{R}^2$. Here δ is the Dirac measure. You may use that

$$\Delta_x v(x, \xi) = -\frac{1}{2\pi}(1 + \ln r) = -\frac{1}{2\pi} + \Phi(x - \xi)$$

where Φ is the fundamental solution for the Laplace operator.

b) Use Green's identity to show that

$$\int_{\partial\Omega} \Big(v \partial_\nu \Delta u - \Delta u \partial_{\nu_x} v \Big) \, dS_x = \int_\Omega \Big(v \Delta^2 u - \Delta u \Delta_x v \Big) \, dx$$

where $u \in C^4(\bar{\Omega})$, and Ω is a bounded subset of $\mathbb{R}^2$ with smooth boundary.

c) Show that

$$
\begin{aligned}
u(\xi) = &- \int_\Omega v(x,\xi)\Delta^2 u(x)\,dx \\
&+ \int_{\partial\Omega} \Big(v(x,\xi)\partial_\nu \Delta u(x) - \Delta u(x)\partial_{\nu_x} v(x,\xi) \\
&\qquad - u(x)\partial_{\nu_x}\Delta_x v(x,\xi) + \Delta_x v(x,\xi)\partial_\nu u(x)\Big)\,dS_x,
\end{aligned}
$$

for $\xi \in \Omega$. You may use that

$$
u(\xi) = -\int_\Omega \Phi(x-\xi)\Delta u(x)\,dx - \int_{\partial\Omega} \Big(u(x)\partial_{\nu_x}\Phi(x-\xi) - \Phi(x-\xi)\partial_\nu u(x)\Big)\,dS_x,
$$

for $\xi \in \Omega$.

d) Assume that we can find for each $\xi \in \Omega$ a function $\varphi_\xi(x)$ with $\Delta_x^2 \varphi_\xi = 0$ for $x \in \Omega$ such that $G(x,\xi) = v(x,\xi) + \varphi_\xi(x)$ satisfies

$$
G(x,\xi) = 0, \quad \partial_{\nu_x} G(x,\xi) = 0, \quad \xi \in \Omega, x \in \partial\Omega.
$$

Assume that

$$
\begin{aligned}
\Delta^2 u &= f \text{ in } \Omega, \\
u = 0, \quad \partial_\nu u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

for $f \in C(\partial\Omega)$. Show that

$$
u(\xi) = -\int_\Omega G(x,\xi) f(x)\,dx.
$$