Solutions
for
TMA4305 Partial Differential Equations

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Problem 1

a) Characteristics are given by

$$\frac{\partial t}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \xi} = z, \quad \frac{\partial x}{\partial \xi} = 0$$

with solution

$$t = \xi + t_0, \quad x = z_0\xi + x_0, \quad z = z_0.$$

For $\xi = 0$ we parametrize the initial data by $t = 0$, $x = \eta$, and $z = u_0(\eta)$, which gives

$$t = \xi, \quad x = u_0(\eta)\xi + \eta, \quad z = u_0(\eta)$$

which are the characteristics.

b) We have

$$u(x,t) = u = z(\eta) = u_0(x - u_0(\eta)t) = u_0(x - ut).$$

c) Given a positive and fixed t. Consider the equation $u = u_0(x - tu)$ as an implicit equation for u in terms of x. We see that

$$\frac{\partial u}{\partial x} = u_0'(x - tu)\big(1 - t\frac{\partial u}{\partial x}\big)$$

or

$$\frac{\partial u}{\partial x} = \frac{u_0'(x - tu)}{1 + tu_0'(x - tu)}$$

which is strictly positive. The implicit function theorem then gives that we can find a solution $u = u(x,t)$.

Problem 2

Let u and v be two solutions. Define $w = u - v$, which then satisfies

$$w_t - \Delta w = 0 \text{ in } U_T, \quad w = 0 \text{ on } \Gamma_T.$$

Define the energy

$$E(t) = \int_U w^2(x,t)dx$$

with derivative

$$\begin{aligned}\dot{E}(t) &= 2\int_U w w_t dx \\ &= 2\int_U w\Delta w dx \\ &= -2\int_U |Dw|^2 dx \leq 0.\end{aligned}$$

Since $E(0) = 0$ and $E(t) \geq 0$, we conclude that $E(t) = 0$ for all $t \in [0, T]$. Thus w is identically zero.

Problem 3

We find

$$\begin{aligned}\frac{d}{dt}\big(k(t) + p(t)\big) &= \int_{\mathbb{R}} \big(u_x u_{xt} + u_t u_{tt}\big) dx \\ &= \int_{\mathbb{R}} \big(-u_{xx} u_t + u_t u_{tt}\big) dx \\ &= \int_{\mathbb{R}} \big(-u_{xx} + u_{tt}\big) u_t dx = 0.\end{aligned}$$

We can write the solution as

$$u(x,t) = F(x-t) + G(x+t)$$

for two smooth, compactly supported functions F and G. This implies

$$\begin{aligned}k(t) &= \int_{\mathbb{R}} \big(F'(x-t) + G'(x+t)\big)^2 dx, \\ p(t) &= \int_{\mathbb{R}} \big(-F'(x-t) + G'(x+t)\big)^2 dx.\end{aligned}$$

Since both F and G have compact support, we have for all sufficiently large times, $t \gg 1$, that

$$\{x \in \mathbb{R} \mid F'(x-t) \neq 0\} \cap \{x \in \mathbb{R} \mid G'(x+t) \neq 0\} = \emptyset.$$

Thus for $t \gg 1$

$$\begin{aligned}k(t) &= \int_{\mathbb{R}} \big(F'(x-t)^2 + G'(x+t)^2\big) dx, \\ p(t) &= \int_{\mathbb{R}} \big((-F'(x-t))^2 + G'(x+t)^2\big) dx,\end{aligned}$$

and hence $k(t) = p(t)$ for all sufficiently large t.

Problem 4 (See Evans p. 101f) The characteristics read

$$\dot{x} = 1, \quad \dot{y} = 1, \quad \dot{z} = z^2,$$

with solution

$$x = x_0 + s, \quad y = s, \quad z = \frac{z_0}{1 - sz_0}.$$

The solution reads

$$u(x, y) = \frac{g(x - y)}{1 - yg(x - y)}$$

which only is well-defined if $1 - yg(x - y) \neq 0$.