Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations

Exam, August 8, 2012, Time: 9:00–13:00.

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Grades will be announced August 29, 2012.

Aids: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Problem 1

a) Consider the equation

$$u_t + uu_x = 0$$

with initial data $u|_{t=0} = u_0$. Determine the characteristics for this equation.

b) Assume that the equation has a solution $u = u(x,t)$. For a given (x,t), the value of the solution u at (x,t), that is, $u = u(x,t)$, is given implicitly by

$$u = u_0(x - tu). \tag{1}$$

Show this.

c) Show that the equation (1) always can be solved if u_0 is strictly increasing.

Problem 2 Let $U_T = U \times (0,T]$ and $\Gamma_T = \bar{U}_T \setminus U_T$ for some T positive and some open and bounded set $U \subset \mathbb{R}^n$ with smooth boundary.

a) Consider the boundary initial value problem

$$u_t - \Delta u = f \text{ in } U_T, \quad u = g \text{ on } \Gamma_T.$$

Here f and g are given functions that are smooth. Show that there can be at most one solution of this problem.

Problem 3 Let $g, h \in C^2(\mathbb{R})$ have compact support. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ be the solution of the wave equation

$$u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty),$$
$$u = g, \quad u_t = h \text{ on } \mathbb{R} \times \{t = 0\}.$$

a) Define

$$k(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x) dx, \quad p(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(t, x) dx.$$

Show that $k(t) + p(t)$ is time independent.

b) Show that

$$k(t) = p(t) \text{ for all sufficiently large } t.$$

Problem 4 Let $U = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and $\Gamma = \mathbb{R} \times \{0\} = \partial U$.

a) Consider the semi-linear equation

$$u_x + u_y = u^2, \text{ in } U,$$
$$u = g, \text{ on } \Gamma.$$

Determine the solution.

b) Is the solution well-defined on U for all g?