TMA4305 Partial Differential Equations

Exam, December 16, 2011

Problem 1

(a) Characteristics are defined by

$$\frac{dx}{dt} = (x+1)^2, \quad \frac{dy}{dt} = (y-1)^2, \quad \frac{dz}{dt} = (x+y)z,$$

and the first two equations integrate to

$$x = -1 - \frac{1}{t+x_0}, \quad y = 1 - \frac{1}{t+y_0}.$$

This implies that

$$\frac{dz}{dt} = (x+y)z = -\Big(\frac{1}{t+x_0} + \frac{1}{t+y_0}\Big)z$$

which integrates to

$$z = \frac{z_0}{(t+x_0)(t+y_0)}.$$

(We also observe that we can write $z = z_0(x+1)(y-1)$.)

(b) Insert $t=0$ and parametrize the initial data by $(s, 0, -(1+s))$. Thus

$$x = -1 - \frac{1}{x_0} = s, \quad y = 1 - \frac{1}{y_0} = 0, \quad z = \frac{z_0}{x_0 y_0} = -(1+s).$$

This implies that $x_0 = -1/(1+s)$, $y_0 = 1$ and $z_0 = 1$, which yields

$$u(x,y) = (x+1)(y-1).$$

Problem 2

Green's identity gives

$$2\pi \cdot 2 = \int_{\partial B} \frac{\partial u}{\partial \nu}(x)dS(x) = \int_B \Delta u(x)dx = \pi,$$

which is a contradiction.

Problem 3

(a) We find, using $u\Delta u = \operatorname{div}(uDu) - Du\cdot Du$ and the divergence theorem

$$\begin{aligned}\frac{d}{dt}e(t) &= \int_U uu_t dx = \int_U u\Delta u dx \\ &= \int_U \big(\operatorname{div}(uDu) - Du \cdot Du\big)dx \\ &= \int_{\partial U} u\frac{\partial u}{\partial \nu}dS - \int_U Du\cdot Du\, dx = -\int_U |Du|^2 dx \le 0.\end{aligned}$$

Since $e(0) = 0$ and $e(t) \geq 0$ by definition, we conclude that $e(t) = 0$ for all t.

(b) Let u_1 and u_2 be two solutions with the same functions f, g, h. Define their difference $u = u_1 - u_2$ and compute the corresponding energy $e(t)$. From the previous calculation we know that $e(t) = 0$ for all t. This implies that u is identically zero, thus $u_1 = u_2$, showing uniqueness of the solution.

Problem 4

Let u denote the solution of

$$u_t = \Delta u, \quad u|_{t=0} = g.$$

Then

$$u(x, t+s) = \int_{\mathbb{R}^n} \Phi(x-y, t+s) g(y)\, dy,$$

but it is also equal to

$$u(x, t+s) = \int_{\mathbb{R}^n} \Phi(x-z, t) u(z, s)\, dz$$

which we can write as

$$\begin{aligned} u(x, t+s) &= \int_{\mathbb{R}^n} \Phi(x-z, t) u(z, s)\, dz \\ &= \int_{\mathbb{R}^n} \Phi(x-z, t) \int_{\mathbb{R}^n} \Phi(z-y, s) g(y)\, dy\, dz \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \Phi(x-z, t) \Phi(z-y, s)\, dz \Big) g(y) dy. \end{aligned}$$

Combining the two expressions for $u(x, t+s)$ we find that

$$\int_{\mathbb{R}^n} \Big(\Phi(x-y, t+s) - \int_{\mathbb{R}^n} \Phi(x-z, t) \Phi(z-y, s)\, dz \Big) g(y)\, dy = 0$$

for all g, which proves the statement.