Solution
TMA4305 Partial Differential Equations

December 3, 2010

Problem 1 (a) The characteristics are given by

$$\frac{dt}{ds} = x, \quad \frac{dx}{ds} = -t, \quad \frac{dz}{ds} = z.$$

The solution is for constants A, B given by

$$z = z_0 e^s, \quad t = A\sin s + B\cos s, \quad x = A\cos s - B\sin s.$$

(b)

$$x(s)^2 + t(s)^2 = A^2 + B^2$$

which is constant.
(c) Let $s = 0$ in the equation for the characteristics. Then

$$z = z_0 = h(\eta), \quad t = B = 0, \quad x = A = \eta$$

with solution

$$z = h(\eta)e^s, \quad t = \eta\sin s, \quad x = \eta\cos s$$

which results in

$$u(x,t) = h(\sqrt{x^2+t^2})\exp(\arctan(t/x)).$$

When we insert this function into the equation, we see that it satisfies the equation and the initial data.

Problem 2 (a) Green's theorem gives

$$0 = \int_\Omega \Delta u\, dx = \int_{\partial\Omega} u\, dS = \int_{\partial\Omega} g\, dS.$$

(b) The mean-value theorem for elliptic functions implies

$$v(0) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v dS = 0.$$

Here $|\partial\Omega|$ denotes the measure of $\partial\Omega$.
(c) We find

$$\nabla u(x) = \int_0^1 \nabla v(tx) dt,$$

and

$$\Delta u(x) = \int_0^1 t\Delta v(tx)dt = 0.$$

For the boundary condition we find (using x as ν at a point $x \in \partial\Omega$)

$$\frac{\partial u}{\partial \nu} = x \cdot \nabla u(x) = \int_0^1 x \cdot \nabla v(tx)dt = \int_0^1 \frac{d}{dt}v(tx)dt = v(x) - v(0) = v(x).$$

(d) Let u_1 and u_2 be two solutions, and define $w = u_1 - u_2$. Then w satisfies

$$\Delta w = 0 \quad \text{in } \Omega \text{ and} \qquad w = 0 \quad \text{on } \partial\Omega.$$

The Green's identity yields

$$\int_\Omega |\nabla w|^2 dx = 0$$

which implies that w is a constant, which indeed satisfies the equation. Thus the solution is unique up to a constant.

Problem 3 (a) We estimate in the standard fashion:

$$\begin{aligned}
|B(u,v)| &\leq 3\int_\Omega |\nabla u|\,|\nabla v|dxdy + \gamma\frac{\pi}{2}\int_\Omega |\nabla u|\,|v|dxdy + 2\int_\Omega |u|\,|v|dxdy \\
&\leq 3\left\|\nabla u\right\|_2 \left\|\nabla v\right\|_2 + \gamma\frac{\pi}{2}\left\|\nabla u\right\|_2 \left\|v\right\|_2 + 2\left\|u\right\|_2 \left\|v\right\|_2 \\
&\leq (3 + \frac{\pi}{2}\gamma + 2)(\left\|\nabla u\right\|_2 + \left\|u\right\|_2)(\left\|\nabla v\right\|_2 + \left\|v\right\|_2) \\
&\leq 2(5 + \frac{\pi}{2}\gamma)\left\|u\right\|_{1,2}\left\|v\right\|_{1,2}
\end{aligned}$$

where $\left\|u\right\|_{1,2} = (\left\|u\right\|_2^2 + \left\|\nabla u\right\|_2^2)^{1/2}$ and we have used $a + b \leq (2(a^2+b^2))^{1/2}$.

As for positivity, we find

$$\begin{aligned}
|B(u,v)| &\geq \int_\Omega |\nabla u|^2 dxdy - \gamma\frac{\pi}{2}\int_\Omega |\nabla u|\,|u|dxdy \\
&\geq \left\|\nabla u\right\|_2^2 - \gamma\frac{\pi}{2}\left\|\nabla u\right\|_2 \left\|u\right\|_2 \\
&\geq (1 - \frac{\gamma\pi}{2C(\Omega)}))\left\|\nabla u\right\|_2^2,
\end{aligned}$$

where $C(\Omega)$ is the constant in Poincaré's inequality. Furthermore,

$$\begin{aligned}
\left\|\nabla u\right\|_2 &= \frac{1}{2}(\left\|\nabla u\right\|_2 + \left\|\nabla u\right\|_2) \\
&\geq \frac{1}{2}(\left\|\nabla u\right\|_2 + \frac{1}{C(\Omega)}\left\|u\right\|_2) \\
&\geq \frac{1}{2}\min\big(\frac{1}{C(\Omega)}, 1\big)\left\|u\right\|_{1,2}
\end{aligned}$$

by Poincaré's inequality. Thus we find

$$|B(u,v)| \geq \frac{1}{2}(1 - \frac{\gamma\pi}{2C(\Omega)}) \min\big(\frac{1}{C(\Omega)}, 1\big) \, \|u\|_{1,2}\,,$$

and we need to choose $\gamma < 2C(\Omega)/\pi$.
(b) Define the functional

$$F(v) = -\int_\Omega f v \, dxdy$$

which is linear and bounded when $f \in L^2(\Omega)$. Lax–Milgram's theorem implies the existence of a $u \in H_0^{1,2}(\Omega)$ such that

$$F(v) = B(u,v),$$

which is equivalent to u being a weak solution of

$$Lu = f \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega$$

when

$$L = \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2}{\partial y^2} - \gamma \arctan(y)\frac{\partial}{\partial x} - (1 + \cos(y))$$