Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4305 Partial Differential Equations

Exam, December 3, 2010. Time: 9:00–13:00.

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Grades will be announced January 3, 2011.

Aids: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Problem 1

a) Determine the characteristics for the differential equation

$$xu_t - tu_x = u.$$

b) Let $\gamma = (x(s), t(s), z(s))$ denote a characteristic curve. Show that $x(s)^2 + t(s)^2$ is constant.

c) Find the solution of the differential equation that satisfies

$$u(x,0) = h(x)$$

for a given function h.

Problem 2

Let $\Omega = \{x \in \mathbb{R}^n \mid |x| < 1\}$.

a) Show that if the Neumann problem

(1)
$$\begin{aligned} \Delta u(x) &= 0, \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) &= g(x), \quad x \in \partial\Omega \end{aligned}$$

has a solution, then

(2)
$$\int_{\partial\Omega} g(x)\,dS = 0.$$

b) Let v be a solution of

$$\begin{aligned} \Delta v(x) &= 0, \quad x \in \Omega, \\ v(x) &= g(x), \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

such that g satisfies (2). Show that $v(0) = 0$.

c) Let v be as above and define

$$u(x) = \int_0^1 \frac{v(tx)}{t} dt.$$

Show that u satisfies (1).

d) Does (1) have a unique solution when (2) holds?

Problem 3 Let $\Omega \subset \mathbb{R}^2$ be a bounded and open set with a smooth boundary, and let $\gamma > 0$. Define the bilinear form by

$$B(u,v) = \int_\Omega \big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \gamma \arctan(y)\frac{\partial u}{\partial x}v + (1+\cos(y))uv\big)\,dxdy$$

for $u, v \in H_0^{1,2}(\Omega)$.

a) Show that B is a bounded bilinear form on $H_0^{1,2}(\Omega)$.

b) Show that B is coercive (or positive) if γ is not too big. Give an upper limit on γ based on the constant in Poincaré's inequality.

c) Define the differential operator by

$$L = \frac{\partial^2}{\partial x^2} + 3\frac{\partial^2}{\partial y^2} - \gamma \arctan(y)\frac{\partial}{\partial x} - 1 - \cos(y)$$

on $H^{1,2}(\Omega)$. Assume that $f \in L^2(\Omega)$. Apply the Lax–Milgram theorem to show that the equation

$$Lu = f \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega$$

has a weak solution $u \in H_0^{1,2}(\Omega)$ for γ sufficiently small.