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TMA4305 PARTIELLE DIFFERENSIALLIGNINGER
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(1) $u_t + u u_x = 0 \quad (-\infty < x < \infty,\ t > 0)$

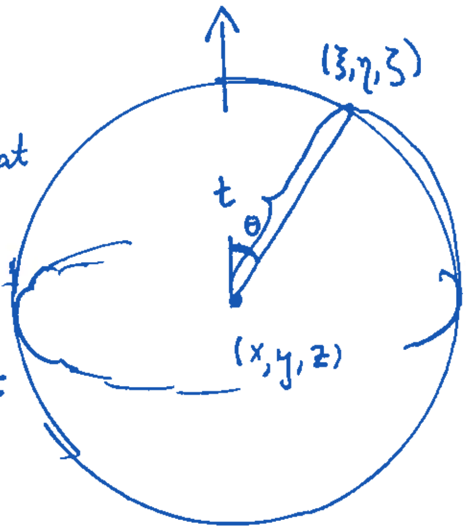
<u>Answer:</u> $u(x,t) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{t+1}, & x \geq 0 \end{cases} \qquad (t > 0)$

No shock!

(2) See the book!

(5) Kirchhoff's formula yields that $u(x,y,z,t)$ is the <u>time</u> <u>derivative</u> <u>of</u>

$$\frac{1}{4\pi t} \iint_{S_t(x,y,z)} (\xi^2 + \eta^2)\, dS_t$$



$$= \frac{1}{4\pi t} \int_{\varphi=0}^{2\pi} \int_0^{\pi} \left[(x + t\sin\theta\cos\varphi)^2 + (y + t\sin\theta\sin\varphi)^2 \right] t^2 \sin\theta \, d\theta\, d\varphi$$

$$= \frac{1}{4\pi t} \left\{ (x^2+y^2)\, 4\pi t^2 + t^4 \cdot \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\varphi}_{2\pi \cdot \frac{4}{3}} \right\}$$

$$= t(x^2+y^2) + \frac{2}{3} t^3, \qquad u(x,y,z,t) = x^2 + y^2 + 2t^2$$

(3) $\mathcal{E}(t) = \frac{1}{2}\iiint\limits_{\Omega}(u_t^2+|\nabla u|^2)\,dx\,dy\,dz$ (Here I write $c=1$.)

$$\mathcal{E}'(t) = \iiint\limits_{\Omega}(u_t u_{tt} + \nabla u\cdot\nabla u_t)\,dx\,dy\,dz$$

Since $u(x,t)\equiv 0$ when $x\in\partial\Omega$, also

$$u_t(x,t) = \lim_{\Delta t\to 0}\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = 0 \text{ for } x\in\partial\Omega.$$

$$0 = \oiint\limits_{\partial\Omega} u_t\nabla u\cdot\vec{n}\,dS \overset{\text{GAUSS}}{=} \iiint\limits_{\Omega}\nabla\cdot(u_t\nabla u)\,dx\,dy\,dz$$

$$= \iiint\limits_{\Omega}(u_t\Delta u + \nabla u\cdot\nabla u_t)\,dx\,dy\,dz$$

$$\mathcal{E}'(t) = \iiint\limits_{\Omega} u_t\underbrace{(u_{tt}-\Delta u)}_{=-u_t}\,dx\,dy\,dz$$

by the eqn. $u_{tt}-\Delta u + u_t = 0$

$$= -\iiint\limits_{\Omega} u_t^2\,dx\,dy\,dz \le 0,$$

since the square $u_t^2 \ge 0$.

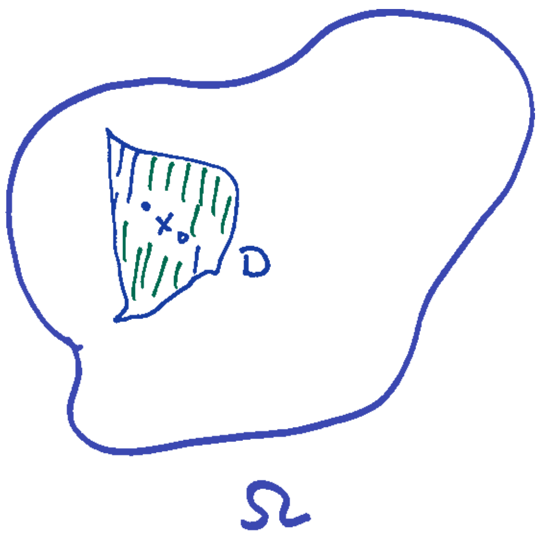
④ $\Delta u = u(u-1)(u+1)$, $u \in C^2(\overline{\Omega})$

ANTITHESIS: $u(x_0) > 1$ at some point $x_0 \in \Omega$. Let $D \subset \Omega$ denote the domain in which $u(x) > 1$ and which contains x_0. On the boundary ∂D, we have $u = 1$. In D

$$\Delta u = u(u^2-1) > 0.$$

Thus u is subharmonic in D and has an interior maximum at x_0, which violates the maximum principle. (Indeed, it is necessary that $\Delta u(x_0) \leq 0$.)

The boundary values $u = 1/2$ on $\partial\Omega$ are needed to conclude that D is strictly comprised in Ω. In other words, $\partial D \cap \partial \Omega$ is empty.



<u>Remark</u>: It is essential that $u \leq 1$ on the boundary $\partial\Omega$. (It was $= 1/2$.)

(6) $$I(v) = \int_0^1\int_0^1 \left(e^x\left(\frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right)dx\,dy.$$

$$I_0 = \inf_v I(v), \quad v - f \in W_0^{1,2}(Q)$$

$0 \le I_0 \le I(f) < \infty$. Choose a <u>minimizing sequence</u> $v_1, v_2, v_3, \ldots$ such that

$$I_0 = \lim_{k\to\infty} I(v_k), \quad I(v_k) < I_0 + 1.$$

Then

$$\int_0^1\int_0^1 |\nabla v_k|^2 dx\,dy \overset{e^x \ge 1}{\le} I(v_k) < I_0 + 1 \quad (k = 1, 2, 3, \ldots)$$

and

$$\|v_k\|_{L^2(Q)} \le \|v_k - f\|_{L^2(Q)} + \|f\|_{L^2(Q)}$$

POINCARÉ

$$\le C_Q \|\nabla(v_k - f)\|_{L^2(Q)} + \|f\|_{L^2(Q)}$$

$$\le C_Q\left(\|\nabla v_k\|_{L^2(Q)} + \|\nabla f\|_{L^2(Q)}\right) + \|f\|_{L^2(Q)}$$

$$\le C_Q\left(\sqrt{I_0 + 1} + \|\nabla f\|_{L^2(Q)}\right) + \|f\|_{L^2(Q)}$$

Thus

$$\|v_k\|_{W^{1,2}(Q)} \le M \quad (k = 1, 2, 3, \ldots)$$

INDEPENDENT OF k

By weak compactness, there exists a function

$u \in W^{1,p}(Q)$ such that

$$\nabla v_{k_j} \longrightarrow \nabla u, \quad v_{k_j} \longrightarrow u \quad \text{weakly in } L^2(Q)$$

at least for a subsequence. Since $v_k - f \in W_0^{1,2}(Q)$, so does $u - f$ (the Sobolev space $W_0^{1,2}(Q)$ is closed under weak convergence.)

By weak lower semicontinuity

$$I(u) \leq \lim I(v_{k_j}) = I_0 .$$

Hence u is a minimizer.

The Euler-Lagrange equation is

$$\boxed{\iint \left(e^x \frac{\partial u}{\partial x} \cdot \frac{\partial \eta}{\partial x} + 2 \frac{\partial u}{\partial y} \cdot \frac{\partial \eta}{\partial y} \right) dx\,dy = 0}$$

Euler-Lagrange (weak form). for all $\eta \in C_0^\infty(Q)$

It follows that

$$\iint \eta \left[\frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2 \frac{\partial u}{\partial y}\right)\right] dx\,dy = 0$$

for all $\eta \in C_0^\infty(Q)$. By the variational lemma

Euler-Lagrange

$$\boxed{\frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2 \frac{\partial u}{\partial y}\right) = 0}$$