Norwegian University of Science and Technology
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Contact during the exam:
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EXAM IN TMA4305 Partial Differential Equations

English
Tuesday, May 27, 2008
15:00 – 19:00

Aids (code C): Approved calculator (HP30S)
Rottman: *Matematisk formelsamling*
One sheet of A4 paper stamped by the IMF, on which you can write what you want.

Results: June 17, 2008

Give arguments for all answers, and include enough computations to show the methods used.

Problem 1 Consider the initial value problem

$$\begin{cases} u_t + 3uu_x = 0 & \text{in} \quad \mathbb{R}\times(0,\infty), \\ u(x,0) = h(x) & \text{in} \quad \mathbb{R}. \end{cases} \tag{1}$$

a) Solve (1) when $h(x) = \frac{5}{3}x - 1$. Sketch the projected characteristics in xt-plane.

b) Find a weak shock solution of (1) when

$$h(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Problem 2 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and consider the boundary value problem

$$(2) \qquad \begin{cases} u_{xx} + 5u_{yy} + b(x,y)u_x = f(x,y) & \text{in } \Omega, \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$ and $b \in C(\bar{\Omega})$.

a) Write down the bilinear form $B(u,v)$ associated to (2).
Use $B(u,v)$ to give a definition of a weak solution of (2).

b) Prove that there exists a unique weak solution to (2) provided

$$\epsilon := 1 - C_\Omega^{1/2}\|b\|_\infty > 0,$$

where C_Ω is the constant in the Poincare inequality.
Hint: Lax-Milgram theorem.

Problem 3 Let $c > 0$ be a constant and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and outward unit normal vector field $\nu = \nu(x,y)$.

a) Let $u \in C^2(\bar{\Omega} \times (0,\infty))$ be a solution of the Neumann problem

$$\begin{cases} u_{tt} + u_t - c^2(u_{xx} + u_{yy}) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

and associate to u the energy function

$$E_u(t) = \frac{1}{2} \iint_\Omega \left(u_t^2 + c^2(u_x^2 + u_y^2) \right) dx\, dy \quad \text{for} \quad t \geq 0.$$

Prove that $\frac{d}{dt} E_u(t) \leq 0$.

b) Let $g, h \in C^2(\bar{\Omega})$ and $f, q \in C^2(\bar{\Omega} \times (0,\infty))$ and consider the following initial boundary value problem

$$(3) \qquad \begin{cases} u_{tt} + u_t - c^2(u_{xx} + u_{yy}) = f(x,y,t) & \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = q(x,y,t) & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g(x,y) \quad \text{and} \quad u_t = h(x,y) & \text{on} \quad \bar{\Omega} \times \{0\}. \end{cases}$$

Prove that solutions of (3) belonging to $C^2(\bar{\Omega} \times [0,\infty))$ are unique.

Problem 4 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $f(x,y) \in L^2(\Omega)$, and define the map $F : W^{1,3}(\Omega) \to \mathbb{R}$ by

$$F(u) = \iint_\Omega \Big(\frac{1}{2}u(u_x^2 + u_y^2) + fu\Big)\,dx\,dy.$$

a) Find the Euler-Lagrange or critical point equation of F in $W_0^{1,3}(\Omega)$.

b) Let $u(x,y) \in C^2(\Omega)$ be a solution of the Euler-Lagrange equation in (a). Show that $u(x,y)$ is a classical solution of

$$u\Delta u + \frac{1}{2}|\nabla u|^2 = f(x,y) \quad \text{in} \quad \Omega.$$

Hint: $C_0^\infty(\Omega) \subset W_0^{1,3}(\Omega)$.

Problem 5 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $u(x,y) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\Delta u + |\nabla u| \geq 0 \quad \text{in} \quad \Omega.$$

Set $w(x,y) := u(x,y) + \epsilon e^{rx}$. Find an $r > 0$ such that

$$\Delta w + |\nabla w| > 0 \quad \text{in} \quad \Omega \qquad \text{for all} \qquad \epsilon > 0.$$

Prove that u satisfies the weak maximum principle, that is prove that

$$\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$