Norwegian University of Science and Technology
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TMA4305 Partial Differential Equations Spring 2007

Solutions for exam questions

1 The principal symbol is $au_{xx}+bu_{xy}+cu_{yy}$ with $a=1$, $b=-2\sin x$ and $c=-\cos^2 x$. Since $b^2-4ac=4\sin^2 x+4\cos^2 x=4>0$, the equation is hyperbolic. The characteristic curves are found by integrating

$$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}=-\sin x\pm 1,$$

so $y=\cos x+x+C$ and $y=\cos x-x+C'$ are the characteristic curves, for arbitrary C, C'.

2 The ODEs for the characteristics are:

$$\frac{dx}{dt}=x^2, \qquad x(0)=s,$$
$$\frac{dy}{dt}=y^2, \qquad y(0)=2s,$$
$$\frac{dz}{dt}=z^2, \qquad z(0)=1.$$

We solve these by separation, obtaining

$$x=\frac{1}{1/s-t}, \qquad y=\frac{1}{1/2s-t}, \qquad z=\frac{1}{1-t}.$$

Next, we solve for t in terms of x, y (we only need t, since s does not appear in the expression for z). We have $1/s-t=1/x$ and $1/2s-t=1/y$. Multiplying the last equation by -2 and adding the two equations, we get $t=-2/y+1/x$, hence

$$\underline{\underline{u(x,y)=z=\frac{1}{1+2/y-1/x}=\frac{xy}{xy+2x-y}}}.$$

3 Let us introduce the linear operator $Lv=v_{xx}+v_{yy}+xv_x+yv_y$, to simplify the notation.

a) Assume v does have a local maximum at some point $(x_0,y_0)\in\Omega$. Then necessarily, $v_x=v_y=0$ and $v_{xx}, v_{yy}\le 0$ at this point. But this implies $Lv(x_0,y_0)\le 0$, so we have a contradiction.

b) Following the hint, we set $v_\varepsilon(x,y)=u(x,y)+\varepsilon x^2$, where $\varepsilon>0$ is arbitrary. Then $Lv_\varepsilon=Lu+\varepsilon L(x^2)=0+\varepsilon 2(1+x^2)>0$, so by part (a) we know that v_ε has no local maximum in Ω. On the other hand, v_ε is by assumption a continuous function on the compact set $\overline{\Omega}$, hence it attains its maximum at some point in $\overline{\Omega}$, and we conclude that this point must be on the boundary $\partial\Omega$. Thus,

$$\max_{\overline{\Omega}} v_\varepsilon=\max_{\partial\Omega} v_\varepsilon.$$

Using this and the fact that $u\le v_\varepsilon\le u+\varepsilon\pi^2$ in $\overline{\Omega}$, we get

$$\max_{\overline{\Omega}} u\le\max_{\overline{\Omega}} v_\varepsilon=\max_{\partial\Omega} v_\varepsilon\le\left(\max_{\partial\Omega} u\right)+\varepsilon\pi^2.$$

Since this holds for all $\varepsilon>0$, we conclude that $\max_{\overline{\Omega}} u\le\max_{\partial\Omega} u$. Since the converse inequality holds trivially, we have proved $\max_{\overline{\Omega}} u=\max_{\partial\Omega} u$, as desired.

4 Let $\chi_{[-1,1]}(x)$ denote the function which equals 1 when $|x| \le 1$, and 0 otherwise. Thus, the initial condition can be stated as $P(x,0) = 10\chi_{[-1,1]}$ and $P_t(x,0) = \chi_{[-1,1]}$. From d'Alembert's formula (with $c = 4$) we have

$$P(x,t) = \frac{10}{2}\left[\chi_{[-1,1]}(x-4t) + \chi_{[-1,1]}(x+4t)\right] + \frac{1}{8}\int_{x-4t}^{x+4t} \chi_{[-1,1]}(s)\,ds.$$

We are asked to find the maximum of $P(10,t)$ for $t > 0$. Setting $x = 10$ we get

$$P(10,t) = 5\left[\chi_{[-1,1]}(10-4t) + \chi_{[-1,1]}(10+4t)\right] + \frac{1}{8}\int_{10-4t}^{10+4t} \chi_{[-1,1]}(s)\,ds.$$

Since $t > 0$, we have $10 + 4t > 1$, so the expression simplifies to

$$P(10,t) = 5\chi_{[-1,1]}(10-4t) + \frac{1}{8}\int_{10-4t}^{1} \chi_{[-1,1]}(s)\,ds.$$

If $10-4t > 1$, i.e., $t < 9/4$, then clearly $P(10,t) = 0$. Next, if $-1 \le 10-4t \le 1$, i.e., $9/4 \le t \le 11/4$, then

$$P(10,t) = 5 + \frac{1}{8}\int_{10-4t}^{1} 1\,ds = 5 + \frac{4t-9}{8}.$$

Finally, if $10-4t < -1$, i.e., $t > 11/4$, then

$$P(10,t) = 0 + \frac{1}{8}\int_{-1}^{1} 1\,ds = \frac{1}{4}.$$

We conclude that

$$P(10,t) = \begin{cases} 0 & \text{if } 0 < t < 9/4, \\ 5 + (4t-9)/8 & \text{if } 9/4 \le t \le 11/4, \\ 1/4 & \text{if } t > 11/4. \end{cases}$$

Thus, the maximum is $5 + 1/4$, attained at $t = 11/4$. Since $5 + 1/4 < 6$, the building survives.

5 Differentiate under the integral sign and use integration by parts to get, for $t > 0$,

$$\begin{aligned} \mathscr{E}'(t) &= \int_\Omega u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \, dx \\ &= \int_\Omega \underbrace{u_t(u_{tt} - c^2\Delta u)}_{=0,\text{ by the equation}} \, dx + \int_{\partial\Omega} u_t(\nabla u \cdot \nu)\, dS \\ &= \int_{\partial\Omega} u_t \frac{\partial u}{\partial \nu}\, dS. \end{aligned}$$

Since $u(x,t) = 0$ for all $x \in \partial\Omega$ and all $t > 0$, it follows that $u_t(x,t) = 0$ for $x \in \partial\Omega$ and $t > 0$. Thus, the last integral above vanishes, and we conclude that $\mathscr{E}'(t) = 0$ for $t > 0$, proving that $\mathscr{E}(t)$ is constant for $t > 0$. Since $\mathscr{E}(t)$ is continuous for all $t \ge 0$, it follows that $\mathscr{E}(t) = \mathscr{E}(0)$ for all $t > 0$.

6 Fix $x \in \Omega$. Recall that

$$G(x,y) = K(x-y) + \omega_x(y),$$

where $\omega_x(y)$ satisfies

$$\begin{cases} \Delta_y \omega_x = 0 & \text{in } \Omega, \\ \omega_x(y) = -K(x-y) & \text{for } y \in \partial\Omega. \end{cases}$$

Thus, $y \mapsto G(x,y)$ is harmonic for $y \in \Omega$, $y \ne x$, and

$$G(x,y) = 0 \qquad \text{for } y \in \partial\Omega. \tag{1}$$

Since ω_x is a bounded function, and since $K(x-y) \to -\infty$ as $y \to x$, we conclude that there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(x)} \subset \Omega$ and

$$G(x,y) < 0 \qquad \text{for } y \in \overline{B_\varepsilon(x)},\ y \neq x. \tag{2}$$

Define $\Omega' = \Omega \setminus \overline{B_\varepsilon(x)}$. The boundary of Ω' is the union of $\partial\Omega$ and $\partial B_\varepsilon(x)$. So the function $y \mapsto G(x,y)$ is harmonic in Ω', with boundary values ≤ 0; to be precise, the boundary value is $= 0$ on $\partial\Omega$ and < 0 on $\partial B_\varepsilon(x)$, by (1) and (2), respectively. So by the weak maximum principle, $G(x,y) \leq 0$ for all $y \in \overline{\Omega'}$. But $y \mapsto G(x,y)$ is not constant in Ω' (again by (1) and (2)), so the strong maximum principle guarantees that there is no interior maximum point. Therefore, $G(x,y) < 0$ for all $y \in \Omega'$, and hence (using again (2)) for all $y \in \Omega$, $y \neq x$.

7 **a)** We calculate

$$\begin{aligned} F(u+v) - F(u) &= \sum_{i=1}^n \frac{1}{2} \langle x_i u_{x_i} + x_i v_{x_i}, x_i u_{x_i} + x_i v_{x_i} \rangle + \langle f, u+v \rangle - \sum_{i=1}^n \frac{1}{2} \langle x_i u_{x_i}, x_i u_{x_i} \rangle - \langle f, u \rangle \\ &= \underline{\underline{\sum_{i=1}^n \langle x_i u_{x_i}, x_i v_{x_i} \rangle + \sum_{i=1}^n \frac{1}{2} \langle x_i v_{x_i}, x_i v_{x_i} \rangle + \langle f, v \rangle}}. \end{aligned}$$

Therefore,

$$\begin{aligned} D_v F(u) &= \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon \sum_{i=1}^n \langle x_i u_{x_i}, x_i v_{x_i} \rangle + \varepsilon^2 \sum_{i=1}^n \frac{1}{2} \langle x_i v_{x_i}, x_i v_{x_i} \rangle + \varepsilon \langle f, v \rangle}{\varepsilon} \\ &= \sum_{i=1}^n \langle x_i u_{x_i}, x_i v_{x_i} \rangle + \langle f, v \rangle. \end{aligned}$$

The Euler-Lagrange equation for F is therefore

$$\underline{\underline{\sum_{i=1}^n \langle x_i u_{x_i}, x_i v_{x_i} \rangle + \langle f, v \rangle = 0 \qquad \text{for all } v \in H_0^1(\Omega).}}$$

b) By the Cauchy-Schwarz inequality, $|\langle f, u \rangle| \leq \|f\|_2 \|u\|_2$, where $\|\cdot\|_2$ denotes the norm on $L^2(\Omega)$. Hence,

$$F(u) \geq \sum_{i=1}^n \frac{1}{2} \langle x_i u_{x_i}, x_i u_{x_i} \rangle - \|f\|_2 \|u\|_2. \tag{3}$$

Using Young's inequality and then the Poincaré inequality, we estimate

$$\begin{aligned} \|f\|_2 \|u\|_2 &\leq \frac{1}{4\varepsilon} \|f\|_2^2 + \varepsilon \|u\|_2^2 \\ &\leq \frac{1}{4\varepsilon} \|f\|_2^2 + \varepsilon C \|\nabla u\|_2^2 \\ &= \frac{1}{4\varepsilon} \|f\|_2^2 + \varepsilon C \int_\Omega \sum_{i=1}^n u_{x_i}^2 \, dx, \end{aligned} \tag{4}$$

where $C > 0$ only depends on Ω. But using the assumption that $x_1, \ldots, x_n \geq 1$ for all $x \in \Omega$, we see that $u_{x_i}^2 \leq x_i^2 u_{x_i}^2$, so (4) implies

$$\|f\|_2 \|u\|_2 \leq \frac{1}{4\varepsilon} \|f\|_2^2 + \varepsilon C \int_\Omega \sum_{i=1}^n x_i^2 u_{x_i}^2 \, dx = \frac{1}{4\varepsilon} \|f\|_2^2 + \varepsilon C \sum_{i=1}^n \langle x_i u_{x_i}, x_i u_{x_i} \rangle.$$

Plugging this estimate into (3) and choosing $\varepsilon = 1/4C$, we obtain

$$F(u) \geq \sum_{i=1}^n \frac{1}{4} \langle x_i u_{x_i}, x_i u_{x_i} \rangle - C \|f\|_2^2 \geq -C \|f\|_2^2.$$

c) The Euler-Lagrange equation from part (c) can be written

$$\int_\Omega \Big(\sum_{i=1}^n x_i^2 u_{x_i} v_{x_i} + f v \Big)\, dx = 0.$$

We need to identify this as the weak formulation of some PDE. So assume u is smooth and $v \in C_0^\infty(\Omega)$. Then integrating by parts, we transform the above equation to (get the derivatives off v)

$$\int_\Omega \Big(\sum_{i=1}^n \big(-2x_i u_{x_i} - x_i^2 u_{x_i x_i}\big) + f \Big) v\, dx = 0.$$

This holds for all test functions v if and only if the following (elliptic) PDE is satisfied:

$$\sum_{i=1}^n \big(x_i^2 u_{x_i x_i} + 2x_i u_{x_i} \big) = f.$$

The boundary condition is $u = 0$, since we restrict to $u \in H_0^1(\Omega)$.