Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4305 Partial Differential Equations**

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Examination date: 12 December 2023

Examination time (from–to): 15:00 -19:00

Permitted examination support material: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he wants. Specific basic calculator allowed. No other aids permitted.

Other information:
The seven problems 1, 2, 3, 4, 5, 6a, and 6b have equal weight.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1 The function

$$u(x,t) = \begin{cases} \left(\frac{x-1}{t+1}\right)^3, & \text{when} \quad x > \xi(t) \\ 0, & \text{when} \quad x < \xi(t) \end{cases}$$

is a weak solution to the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{3}{4}u^{4/3}\right) = 0$$

in the half-plane $t > 0$, $-\infty < x < \infty$. The unknown shock curve $x = \xi(t)$ starts at the origin. Determine this shock curve and draw a picture with characteristic curves in the $x\,t$-plane.

Problem 2 Let Ω be a bounded domain in $\mathbb{R}^3$ and assume that $w \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to the problem

$$\begin{cases} \Delta w = w^2(w-5) \quad \text{in} \quad \Omega \\ w \leq 8 \quad \text{on the boundary} \quad \partial\Omega. \end{cases}$$

Prove that $w \leq 8$ in Ω. (The notation means that w is continuous up to the boundary and has continuous second derivatives at interior points.)

Problem 3 Let $V = V(x,y,z,t)$ be the solution in $\mathbb{R}^3 \times (0,\infty)$ of the wave equation

$$\frac{\partial^2 V}{\partial t^2} = c^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

with the initial data

$$V(x,y,z,0) = x^2 + y^2 + z^2, \quad V_t(x,y,z,0) = z.$$

Calculate $V(0,0,0,t)$ when $t > 0$.

Problem 4 In $\mathbb{R}^3$ the function

$$\Phi(\mathbf{x}) = \frac{e^{-k|\mathbf{x}|}}{4\pi|\mathbf{x}|} \qquad (k > 0,\ \mathbf{x} = (x_1, x_2, x_3))$$

satisfies the equation

$$(-\Delta + k^2)\Phi(\mathbf{x}) = \delta(\mathbf{x})$$

in the sense of distributions. (You do not have to verify this.) Here δ is Dirac's delta. Which partial differential equation does

$$v(\mathbf{x}) = \iiint\limits_{\mathbb{R}^3} \Phi(\mathbf{x}-\mathbf{y}) \frac{d^3\mathbf{y}}{1+|\mathbf{y}|^2}$$

satisfy? Now you only need to give the correct equation.

Problem 5 Show that the problem

$$\begin{cases} u_{tt} - c^2\Delta u + 5u = 2023 \quad \text{in} \quad \Omega_T = \Omega \times (0,T) \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \times [0,T] \\ u(\mathbf{x},0) = f(\mathbf{x}) \quad \text{when} \quad x \in \Omega \\ u_t(\mathbf{x},0) = g(\mathbf{x}) \quad \text{when} \quad x \in \Omega \end{cases}$$

can have at most one solution $u \in C^2(\overline{\Omega_T})$. Here Ω is a smooth bounded domain in $\mathbb{R}^3$ with the outer unit normal $\mathbf{n}$. (*Hint:* Use the "energy"

$$e(t) = \frac{1}{2}\iiint\limits_{\Omega} \left(w_t^2 + c^2|\nabla w|^2 + 5w^2\right) d^3\mathbf{x}$$

for some suitable w.)

Problem 6 Consider the variational integral

$$I(v) = \int_0^1\int_0^1 \left(e^x\left(\frac{\partial v}{\partial x}\right)^2 + (1+y^2)\left(\frac{\partial v}{\partial y}\right)^2 - e^{xy}v\right) dx\, dy$$

among all functions $v \in H_0^1(\Omega)$. Here Ω is the square $0 < x < 1$, $0 < y < 1$ and $H_0^1(\Omega)$ is a Sobolev space.

a) Prove that

$$-\infty < \inf I(v) \le 0,$$

where the infimum is taken among all $v \in H_0^1(\Omega)$.

b) Assume that the infimum is attained for a minimizing function $u \in H_0^1(\Omega)$. Which *second order* differential equation does u satisfy? In other words, derive the Euler-Lagrange equation. For simplicity, you may now assume that u has continuous second derivatives.

GOOD LUCK!