INFINITE INSULATED THREAD

HEAT EQN.

$$\widehat{f*g} = \sqrt{2\pi}\,\hat{f}\cdot\hat{g}$$

$$\mathcal{F}\{f'\} = i\omega\,\mathcal{F}\{f\}$$

$$\begin{cases} \boxed{u_t = k\,u_{xx}} & (-\infty < x < \infty,\ t > 0) \\ \lim\limits_{t\to 0+} u(x,t) = f(x), & \text{initial temperature} \end{cases}$$

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$$

trans. with respect to x
(t is dumb, a parameter only!)

$$\hat{u}(\omega,0) = \hat{f}(\omega) \qquad \text{OBS!}$$

$$\underline{\underline{\widehat{u_t(x,t)}}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t}\, e^{-i\omega x}\, dx$$

$$= \frac{\partial}{\partial t}\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx \right\} = \underline{\underline{\frac{\partial}{\partial t}\hat{u}(\omega,t)}}$$

$$\widehat{u_{xx}(x,t)} = i^2\omega^2\,\hat{u}(\omega,t) \qquad (i^2 = -1)$$

The Eqn becomes

$$\boxed{\frac{\partial}{\partial t}\hat{u}(\omega,t) + k\omega^2\,\hat{u}(\omega,t) = 0}$$

SOLUTION(S):

$\frac{\partial A}{\partial t} = 0$

$$\hat{u}(\omega, t) = A(\omega)\, e^{-\omega^2 k t}.$$

$$\hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega) \cdot 1$$

$$\hat{u}(\omega, t) = \hat{f}(\omega)\, e^{-\omega^2 k t}$$

$$= \frac{1}{\sqrt{2kt}} \hat{f}(\omega) \cdot \widehat{e^{-x^2/4kt}}$$

$$= \frac{1}{\sqrt{4k\pi t}} \widehat{f(x) * e^{-x^2/4kt}}$$

$$\boxed{u(x, t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\, dy}$$

Recall the Fourier Transform

$$\widehat{e^{-a x^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}.$$

Use $a = \frac{1}{4kt} > 0$

REMARK:

This is the only solution that remains bounded from below for all times t, for instance

$$u(x, t) \geq -273^\circ{,}15, \quad t \geq 0.$$

There are "non-physical" solutions too.