NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4305 Partial Differential Equations

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Examination date: 2 December 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: C: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he wants. Specific basic calculator allowed. No other aids permitted.

Other information:
The eleven problems 1a, 1b, 2a, 2b, 2c, 3a, 3b, 4, 5, 6a, and 6b have equal weight.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
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Date Signature

Problem 1 Consider the differential equation

$$u_t + uu_x = 0. \tag{1}$$

a) Find the solution of (1) with initial data

$$u(0,x) = \begin{cases} 1, & \text{for } x \le 0, \\ 0, & \text{for } x > 0. \end{cases}$$

Show that the solution is a weak solution.

b) Consider now equation (1) with the initial condition

$$u(0,x) = \begin{cases} 0, & \text{for } x \le 0, \\ 1, & \text{for } x > 0. \end{cases}$$

Show that

$$u(t,x) = \begin{cases} 0, & \text{for } x \le 0, \\ \frac{x}{t}, & \text{for } 0 < x < t, \\ 1, & \text{for } x \ge t, \end{cases} \tag{2}$$

is a weak solution of (1).

Problem 2

Consider the differential equation

$$u_t - u_{xx} = 0. \tag{3}$$

a) We will consider solutions of the differential equation of the form $u(t,x) = U(x/\sqrt{t})$. Show that U satisfies

$$-\frac{\xi}{2}U'(\xi) = U''(\xi). \tag{4}$$

b) Solve (4) and express the solution in terms of the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

c) Find the solution of the differential equation (3) such that

$$u(t,0) = 1, t > 0, \quad u(0,x) = 0,\ x > 0.$$

You may use that $\operatorname{erf}(x) \to 1$ as $x \to \infty$.

Problem 3 Consider the the problem of minimizing the variational integral

$$I(v) = \int_\Omega \left(|\nabla v|^2 + 5v^2 \right) d^3\mathbf{x}$$

among all functions $v \in H^1(\Omega) \cap C(\overline{\Omega})$ with boundary values $v|_{\partial\Omega} = 1$. Here Ω is a bounded domain in $\mathbb{R}^3$.

a) Derive the Euler-Lagrange equation.

b) Assume that there exists a minimizer $v \in C^2(\overline{\Omega})$. Show that $0 \le v \le 1$.

Problem 4 Suppose that function $w = w(\mathbf{x}, t)$ belonging to $C^2(\overline{\Omega} \times [0, \infty))$ is a solution to

$$\frac{\partial w}{\partial t} = \Delta w \quad \text{in} \quad \Omega \times [0, \infty).$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$. Assume that $w(\mathbf{x}, t) = 0$ when $\mathbf{x} \in \partial\Omega$ and $t > 0$. Show that

$$\int_\Omega w(\mathbf{x}, t)^2 \, d^n\mathbf{x} \le \int_\Omega w(\mathbf{x}, 0)^2 \, d^n\mathbf{x} \quad \text{when} \quad t > 0.$$

Problem 5 Show that

$$\Phi(x) = \frac{1}{2k} e^{-k|x|}, \qquad x \in \mathbb{R},$$

is a solution of the equation

$$-\frac{d^2\Phi}{dx^2} + k^2 \Phi = \delta$$

in the sense of distributions. Then find a solution to

$$-v'' + k^2 v = f, \quad \text{where} \quad f \in C^\infty(\mathbb{R}) \quad \text{has compact support.}$$

Problem 6 Consider the wave equation

$$u_{tt} - u_{xx} = \sin(x). \tag{5}$$

a) Show that $u_0(t,x) = -\sin(x)(\cos(t) - 1)$ satisfies (5).

b) Consider equation (5) with initial data

$$u(0,x) = g(x), \quad u_t(0,x) = h(x), \quad x \in (0,\pi), \tag{6}$$

for given functions g, h, and boundary values

$$u(t,0) = u(t,\pi) = 0, \quad t > 0. \tag{7}$$

Find the general solution of (5) with initial and boundary values (6), (7). Explain how the solution you find is defined for all times.