Problem 1

(a) From Theorem 10.6 we find that the solution reads
\[ u(0, x) = \begin{cases} 
1, & \text{for } x \leq t/2, \\
0, & \text{for } x > t/2, 
\end{cases} \]
using the Rankine–Hugoniot relation with flux function \( q(u) = u^2 / 2 \). Furthermore, the theorem says it is weak solution.

(b) We need to show that
\[ 0 = \int_0^{\infty} \int_{-\infty}^{\infty} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx 
\]
for all compactly supported test functions \( \phi \). To that end we find
\[
\begin{align*}
\int_0^{\infty} \int_{-\infty}^{\infty} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt &+ \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= \int_0^{\infty} \left( \int_{-\infty}^{0} + \int_{0}^{t} + \int_{t}^{\infty} \right) (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= -\int_{-\infty}^{0} \phi(0, x) dx + \int_0^{\infty} \int_{-\infty}^{0} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= -\int_{-\infty}^{0} \phi(0, x) dx + \int_0^{\infty} \int_{-\infty}^{t} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&\quad + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= -\int_{-\infty}^{0} \phi(0, x) dx + \int_0^{\infty} \int_{-\infty}^{t} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&\quad + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= -\int_{-\infty}^{0} \phi(0, x) dx + \int_0^{\infty} \int_{-\infty}^{t} (u \phi_t + \frac{1}{2} u^2 \phi_x) dx \, dt + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&\quad + \int_{-\infty}^{\infty} u_0(x) \phi(0, x) dx \\
&= 0.
\end{align*}
\]

Here we have first divided the domain of integration according to the definition of \( u \). The integral for \( x < 0 \) can be computed by integrating \( \phi_t \) with respect to time, giving only contribution from \( t = 0 \). The integral for \( x > t \) vanishes since \( u = 0 \) here. Thus we are left with the integral for \( 0 < x < t \) in the \( x \)-direction. Here we use that \( u \) satisfies the equation pointwise, and we can add \((u_t + (u^2)_x)/2\) \( \phi = 0 \), and then use Leibniz formula. By applying Gauss’s (or Green’s) theorem we convert this to an integral over the boundary of the domain, with \( n \) denoting the outward unit normal. Here \( T \) is such that \( \phi(t, x) = 0 \) for
all \( t \geq T \) and all \( x \). However, here \( u \) or \( \phi \) vanishes, and thus the integral is zero. We are left with integrals at \( t = 0 \), and using the form of \( u_0 \) also this vanishes.

**Problem 2**

(a) Let \( \xi = x/\sqrt{t} \). Direction differentiation gives

\[
- \frac{x}{2t^{3/2}} U'(\xi) = \frac{1}{t} U''(\xi),
\]
or

\[
- \frac{\xi}{2} U'(\xi) = U''(\xi).
\]

(b) Introduce \( V = U' \) and integrate

\[ V = V_0 e^{-\xi^2/4}. \]

Integrate to get \( U \), thus

\[ U(\xi) = V_0 \int_0^\xi e^{-\eta^2/4} d\eta + U_0 = 2V_0 \int_0^{\xi/2} e^{-z^2} dz + U_0 = \sqrt{\pi} V_0 \text{erf}\left(\frac{\xi}{2}\right) + U_0. \]

In terms of \( u \) we find

\[ u(t, x) = U(\frac{x}{\sqrt{t}}) = \sqrt{\pi} V_0 \text{erf}\left(\frac{x}{2\sqrt{t}}\right) + U_0. \]

(c) By inserting \( x = 0 \) we find that \( U_0 = 1 \), and by letting \( t \to 0 \) we find that \( V_0 = -1/\sqrt{\pi} \). Thus

\[ u(t, x) = - \text{erf}\left(\frac{x}{2\sqrt{t}}\right) + 1. \]

**Problem 3a** Let \( \phi \) be a function in \( C^\infty_0(\Omega) \) and consider

\[ I(v + t\phi) = I(v) + 2t \int_\Omega (\nabla v \cdot \nabla \phi + 5v \phi) \, d^3x + O(t^2). \]

The necessary condition

\[ \frac{dI(v + t\phi)}{dt} \bigg|_{t=0} = 0 \]

yields the Euler-Lagrange equation

\[ \int_\Omega (\nabla v \cdot \nabla \phi + 5v \phi) \, d^3x \]

in weak form.

**Problem 3b.** Integrating by parts we get

\[ \int_\Omega \phi(\Delta v + v) \, d^3x = 0 \]
for all test-functions. By “the variational lemma” we have pointwise
\[ \Delta v = 5v. \]
If \( v > 1 \) at some point, then there is an interior maximum:
\[ v(x_0) = \max_{\Omega} v > 1 \]
But this yields the contradiction
\[ 0 \geq \Delta v(x_0) = 5v(x_0) > 5. \]
If \( v < 0 \) at some point, then there is an interior minimum:
\[ 0 > v(y_0) = \min_{\Omega} v \]
This yields the contradiction
\[ 0 \leq \Delta v(y_0) = 5v(y_0) < 0. \]
It follows that \( 0 \leq v \leq 1. \)

**Problem 4** Differentiate with respect to \( t \) to see that
\[
\frac{d}{dt} \int_{\Omega} w(x, t)^2 \, d^n x = 2 \int_{\Omega} w w_t \, d^n x = 2 \int_{\Omega} w \Delta w \, d^n x
\]
\[= -2 \int_{\Omega} |\nabla w|^2 \, d^n x + \oint w \nabla w \cdot n \, dS = -2 \int_{\Omega} |\nabla w|^2 \, d^n x + 0 \leq 0. \]
Thus the integral is decreasing in \( t \) and the inequality follows. We used Gauss’s theorem (the divergence thm) on
\[ \text{div}(w \nabla w) = \nabla w \cdot \nabla w + w \Delta w. \]

**Problem 5**
We have to verify that
\[ \int_{-\infty}^{+\infty} e^{-k|x|} (-\phi''(x) + k^2 \phi(x)) \, dx = 2k \phi(0) \]
for all test functions \( \phi \). Integrations by parts yield
\[ - \int_{-\infty}^{0} e^{+kx} \phi''(x) \, dx = -\phi'(0) + k\phi(0) - k^2 \int_{-\infty}^{0} e^{+kx} \phi(x) \, dx \]
\[ - \int_{0}^{\infty} e^{-kx} \phi''(x) \, dx = +\phi'(0) + k\phi(0) - k^2 \int_{0}^{\infty} e^{-kx} \phi(x) \, dx. \]
Adding the two previous identities we arrive at the desired identity
\[-\int_{-\infty}^{+\infty} e^{-k|x|} \phi''(x) \, dx = 2k \phi(0) - k^2 \int_{-\infty}^{+\infty} e^{-k|x|} \phi(x) \, dx.\]

We know that a solution of \(-v'' + k^2 v = f(x)\) is given by the convolution
\[v(x) = (\Phi * f)(x) = \frac{1}{2k} \int_{-\infty}^{+\infty} f(y)e^{-k|x-y|} \, dy.\]

**Problem 6**

(a) We have
\[u_{0,tt} = \sin(x) \cos(t), \quad u_{0,xx} = \sin(x)(\cos(t) - 1),\]
showing that the given function is a solution of the equation.

(b) Consider \(v = u - u_0\). Then \(v\) satisfies
\[v_{tt} - v_{xx} = 0. \quad (1)\]

The initial and boundary conditions yield in the standard manner the solution, using d’Alembert’s formula,
\[v(t, x) = \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(z) \, dz \quad (2)\]
where \(g\) and \(h\) are extended to all real arguments by
\[g(-x) = -g(x), h(-x) = -h(x), \quad g(x + 2\pi) = g(x), h(x + 2\pi) = h(x). \quad (3)\]

The solution then reads
\[u(t, x) = v(t, x) + u_0(t, x)\]
\[= \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(z) \, dz - \sin(x)(\cos(t) - 1).\]