Problem 1

(a) From Theorem 10.6 we find that the solution reads

$$u(0,x) = \begin{cases} 1, & \text{for } x \le t/2, \\ 0, & \text{for } x > t/2, \end{cases}$$

using the Rankine–Hugoniot relation with flux function $q(u) = u^2/2$. Furthermore, the theorem says it is weak solution.

(b) We need to show that

$$0 = \int_0^\infty \int_{-\infty}^\infty \big(u\phi_t + \frac{1}{2}u^2\phi_x\big)dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx$$

for all compactly supported test functions ϕ. To that end we find

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty \big(u\phi_t + \frac{1}{2}u^2\phi_x\big)dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= \int_0^\infty \Big(\int_{-\infty}^0 + \int_0^t + \int_t^\infty\Big)\big(u\phi_t + \frac{1}{2}u^2\phi_x\big)dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= -\int_{-\infty}^0 \phi(0,x)dx + \int_0^\infty \int_0^t \big(u\phi_t + \frac{1}{2}u^2\phi_x\big)dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= -\int_{-\infty}^0 \phi(0,x)dx + \int_0^\infty \int_0^t \big(u\phi_t + \frac{1}{2}u^2\phi_x + (u_t + \frac{1}{2}(u^2)_x)\phi\big)dx\,dt \\
&\quad + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= -\int_{-\infty}^0 \phi(0,x)dx + \int_0^\infty \int_0^t \big((u\phi)_t + \frac{1}{2}(u^2\phi)_x\big)dx\,dt + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= -\int_{-\infty}^0 \phi(0,x)dx + \int_{\{x=0\}\cup\{x=t\}\cup\{t=T\}} \big(u\phi, \frac{1}{2}u^2\phi\big)\cdot n\,dS \\
&\quad + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= -\int_{-\infty}^0 \phi(0,x)dx + \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \\
&= 0.
\end{aligned}$$

Here we have first divided the domain of integration according to the definition of u. The integral for $x < 0$ can be computed by integrating ϕ_t with respect to time, giving only contribution from $t = 0$. The integral for $x > t$ vanishes since $u = 0$ here. Thus we are left with the integral for $0 < x < t$ in the x-direction. Here we use that u satisfies the equation pointwise, and we can add $(u_t + (u^2)_x/2)\phi = 0$, and then use Leibniz formula. By applying Gauss's (or Green's) theorem we convert this to an integral over the boundary of the domain, with n denoting the outward unit normal. Here T is such that $\phi(t,x) = 0$ for all $t \geq T$ and all x. However, here u or ϕ vanishes, and thus the integral is zero. We are left with integrals at $t = 0$, and using the form of u_0 also this vanishes.

Problem 2

(a) Let $\xi = x/\sqrt{t}$. Direction differentiation gives

$$-\frac{x}{2t^{3/2}} U'(\xi) = \frac{1}{t} U''(\xi),$$

or

$$-\frac{\xi}{2} U'(\xi) = U''(\xi).$$

(b) Introduce $V = U'$ and integrate

$$V = V_0 e^{-\xi^2/4}.$$

Integrate to get U, thus

$$U(\xi) = V_0 \int_0^{\xi} e^{-\eta^2/4} d\eta + U_0 = 2V_0 \int_0^{\xi/2} e^{-z^2} dz + U_0 = \sqrt{\pi} V_0 \operatorname{erf}\big(\frac{\xi}{2}\big) + U_0.$$

In terms of u we find

$$u(t,x) = U(\frac{x}{\sqrt{t}}) = \sqrt{\pi} V_0 \operatorname{erf}\big(\frac{x}{2\sqrt{t}}\big) + U_0.$$

(c) By inserting $x = 0$ we find that $U_0 = 1$, and by letting $t \to 0$ we find that $V_0 = -1/\sqrt{\pi}$. Thus

$$u(t,x) = -\operatorname{erf}\big(\frac{x}{2\sqrt{t}}\big) + 1.$$

Problem 3a Let ϕ be a function in $C_0^{\infty}(\Omega)$ and consider

$$I(v + t\phi) \; = I(v) + 2t \int_{\Omega} (\nabla v \cdot \nabla \phi + 5v\, \phi)\, d^3\mathbf{x} + O(t^2).$$

The necessary condition

$$\frac{dI(v + t\phi)}{dt}\Big|_{t=0} \; = \; 0$$

yields the Euler-Lagrange equation

$$\int_{\Omega} (\nabla v \cdot \nabla \phi + 5v\phi)\, d^3\mathbf{x}$$

in weak form.

Problem3b. Integrating by parts we get

$$\int_{\Omega} \phi(\Delta v + v)\, d^3\mathbf{x} \; = \; 0$$

for all test-functions. By "the variational lemma" we have pointwise

$$\Delta v \;=\; 5v.$$

If $v > 1$ at some point, then there is an interior maximum:

$$v(\mathbf{x}_0) = \max_{\overline{\Omega}} v > 1$$

But this yields the contradiction

$$0 \geq \Delta v(\mathbf{x}_0) = 5v(\mathbf{x}_0) > 5.$$

If $v < 0$ at some point, then there is an interior minimum:

$$0 > v(\mathbf{y}_0) = \min_{\overline{\Omega}} v$$

This yields the contradiction

$$0 \leq \Delta v(\mathbf{y}_0) = 5v(\mathbf{y}_0) < 0.$$

It follows that $0 \leq v \leq 1$.

Problem 4 Differentiate with respect to t to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega w(\mathbf{x},t)^2\,d^n\mathbf{x} \;&=\; 2\int_\Omega w\,w_t\,d^nx = 2\int_\Omega w\Delta w\,d^nx \\ = -2\int_\Omega |\nabla w|^2\,d^n\mathbf{x} \;&+\; \oint w\nabla w\cdot\mathbf{n}\,dS = -2\int_\Omega |\nabla w|^2\,d^n\mathbf{x} + 0 \leq 0. \end{aligned}$$

Thus the integral is decreasing in t and the inequality follows. We used Gauss's theorem (the divergence thm) on

$$\operatorname{div}(w\nabla w) \;=\; \nabla w\cdot\nabla w + w\Delta w.$$

Problem 5
We have to verify that

$$\int_{-\infty}^{+\infty} e^{-k|x|}\big(-\phi''(x) + k^2\phi(x)\big)\,dx \;=\; 2k\,\phi(0)$$

for all test functions ϕ. Integrations by parts yield

$$\begin{aligned} -\int_{-\infty}^{0} e^{+kx}\phi''(x)\,dx \;&=\; -\phi'(0) + k\phi(0) - k^2\int_{-\infty}^{0} e^{+kx}\phi(x)\,dx \\ -\int_{0}^{\infty} e^{-kx}\phi''(x)\,dx \;&=\; +\phi'(0) + k\phi(0) - k^2\int_{0}^{\infty} e^{-kx}\phi(x)\,dx. \end{aligned}$$

Adding the two previous identities we arrive at the desired identity

$$-\int_{-\infty}^{+\infty} e^{-k|x|}\phi''(x)\,dx \;=\; 2k\,\phi(0) - k^2\int_{-\infty}^{+\infty} e^{-k|x|}\phi(x)\,dx.$$

We know that a solution of $-v'' + k^2 v \;=\; f(x)$ is given by the convolution

$$v(x) \;=\; (\Phi * f)(x) \;=\; \frac{1}{2k}\int_{-\infty}^{+\infty} f(y)e^{-k|x-y|}\,dy.$$

Problem 6

(a) We have

$$u_{0,tt} = \sin(x)\cos(t), \quad u_{0,xx} = \sin(x)(\cos(t) - 1),$$

showing that the given function is a solution of the equation.

(b) Consider $v = u - u_0$. Then v satisfies

$$v_{tt} - v_{xx} = 0. \tag{1}$$

The initial and boundary conditions yield in the standard manner the solution, using d'Alembert's formula,

$$v(t,x) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(z)dz \tag{2}$$

where g and h are extended to all real arguments by

$$g(-x) = -g(x), h(-x) = -h(x), \quad g(x+2\pi) = g(x), h(x+2\pi) = h(x). \tag{3}$$

The solution then reads

$$\begin{aligned} u(t,x) &= v(t,x) + u_0(t,x) \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(z)dz - \sin(x)(\cos(t) - 1). \end{aligned}$$