ELLIPTIC 2nd ORDER EQS

$$\boxed{L(u) \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + \cancel{c(x)u} = f(x)}$$

$(x = (x_1, x_2, \dots, x_n) \in \Omega)$

DRIFT or CONVECTION TERM

Keep $c=0$! for simplicity.

$\mathrm{Trace}(AD) \geq 0.$

We assume

Can always be arranged!

1) $a_{ij} = a_{ji}$ (symmetry)

It follows that $a_{11}(x) \geq \gamma$.

2) $$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \gamma > 0,$$

for all $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ (uniform ELLIPTICITY)

3) bounded coefficients (needed $b_1(x) \geq -M$)

The following problems appear:

- Existence of a solution with given boundary data (Dirichlet or Neumann cond.)
- Uniqueness
- Stability
- Maximum/minimum Principles
- Regularity of solutions (e.g. differentiability)
- Boundary behaviour
- Etc.

Trace A
$= a_{11} + a_{22} + \cdots + a_{nn}$

In matrix notation

- $$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \text{Trace}(A\mathbb{D})$$

where $A = (a_{ij})$, $\mathbb{D} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ are symmetric $n \times n$ - matrices: $A = A^T$, $D = D^T$ (transposes).

Preliminaries

Trace(AB) = Trace(BA) for square matrices.

DEF: $A \geq 0 \iff$

$$\langle A\xi, \xi\rangle = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \text{ for all}$$

$\xi = (\xi_1, \xi_2, \ldots, \xi_n)$

We say that the quadratic form is positive semi-definite.

LEMMA For symmetric matrices
$A \geq 0$ & $B \geq 0 \implies \text{Trace}(AB) \geq 0$.

~~PROOF:~~
$AB = AB^{1/2}B^{1/2}$ $(B = B^T)$

Trace(AB) = Trace$(B^{1/2}AB^{1/2})$

$$\langle B^{1/2}AB^{1/2}\xi, \xi\rangle = \langle A(B^{1/2}\xi), (B^{1/2}\xi)\rangle$$

$$= \langle A\zeta, \zeta\rangle \geq 0 \quad \text{where } \zeta = B^{1/2}\xi \quad \text{for all } \xi.$$

Thus $B^{1/2}AB^{1/2} \geq 0$ by definition and hence Trace$(B^{1/2}AB^{1/2}) \geq 0$ (Remark: All diagonal elements are ≥ 0 for a pos. semi-definite matrix.)
The trace is their sum.

Assume now that $u \in C^2(\Omega)$

LEMMA At an interior maximum point of $u = u(x)$, we have $L(u) \leq 0$; at an interior minimum point $L(u) \geq 0$.

<u>Proof</u> for minimum. By assumption $A \geq 0$ and by the infinitesimal calculus $\mathbb{D} \geq 0$ at the minimum point. At the minimum also $\nabla u = 0$. Thus

$$L(u) = \mathrm{Trace}(A\mathbb{D}) + 0 \geq 0$$

at the minimum point (by previous lemma).

THEOREM Assume that $u \in C^2(\Omega)$, $u \in C(\bar{\Omega})$, where Ω is a <u>bounded</u> domain. If $L(u(x)) \leq 0$ in Ω,

SUPERSOLUTION

<u>Ex.</u> $\Delta u \leq 0$

SUPERHARMONIC

then

$$\min_{x \in \bar{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$$

<u>Proof:</u> At an interior minimum point the auxiliary function

$$u_\varepsilon(x) = u(x) - \varepsilon e^{\alpha x_1}$$

satisfies

BY LEMMA

$$0 \le L(u_\varepsilon) = L(u) - \varepsilon a_{11}\alpha^2 e^{\alpha x_1} - b_1 \varepsilon\alpha e^{\alpha x_1}$$

$$\le 0 - \varepsilon\alpha e^{\alpha x_1}(\alpha a_{11} + b_1)$$

$$\le -\varepsilon\alpha e^{\alpha x_1}(\alpha\gamma - \inf b_1)$$

$$\le -\varepsilon\alpha e^{\alpha \min_\Omega (x_1)}(\alpha\gamma - \inf b_1)$$

$$< 0 \quad \text{if} \quad \alpha > \frac{\inf b_1}{\gamma}, \quad \alpha > 0.$$

This contradiction shows that u_ε cannot have interior minima. Hence

$$u(x) - \varepsilon e^{\alpha x_1} \ge \min_{x\in\partial\Omega}\left(u(x) - \varepsilon e^{\alpha x_1}\right)$$

$$\ge \min_{x\in\partial\Omega} u(x) - \varepsilon e^{\alpha \max(x_1)}$$

Let $\varepsilon \to 0$. It follows that

$$u(x) \ge \min_{\partial\Omega} u. \qquad \square$$

The maximum principle for $L(u) \ge 0$ is similar.

THEOREM The solution $u \in C(\bar\Omega) \cap C^2(\Omega)$ to the equation $L(u) = f$ with boundary values $u|_{\partial\Omega} = g$ is unique in the bounded domain Ω (if it exists).

Proof: $u = u_2 - u_1$, $L(u) = 0$, $u|_{\partial\Omega} = 0$. $\square$