ELLIPSTIC 2nd ORDER EQUATIONS

\[ L(u) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + c(x) u = f(x) \]

\((x = (x_1, x_2, ..., x_n) \in \Omega)\)

DRIFT
CONVECTION TERM

Keep \(c = 0\) for simplicity.

True \((A_0) \geq 0\).

We assume can always be arranged:

1) \(a_{ij} = a_{ji}\) (symmetry)

2) \[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma |\xi|^2, \quad \gamma > 0, \]

for all \(\xi = (\xi_1, \xi_2, ..., \xi_n)\) (uniform ELLIPTICITY)

3) bounded coefficients (needed \(b_j(x) \geq -M\))

The following problems appear:

- Existence of a solution with given boundary data (Dirichlet or Neumann cond.)
- Uniqueness
- Stability
- Maximum/minimum Principle
- Regularity of solutions (e.g. differentiability)
- Boundary behaviour
- Etc.
In matrix notation

\[ \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} = \text{Trace} (\mathbf{A} \mathbf{D}) \]

where \( \mathbf{A} = (a_{ij}) \), \( \mathbf{D} = \left( \frac{\partial^2}{\partial x_i \partial x_j} \right) \) are symmetric 
\( n \times n \) matrices: \( \mathbf{A} = \mathbf{A}^T \), \( \mathbf{D} = \mathbf{D}^T \) (transposes).

**Preliminaries**

**Trace (AB) = Trace (BA) for square matrices.**

**DEF:** \( \mathbf{A} \geq 0 \iff \langle \mathbf{A} \overrightarrow{x}, \overrightarrow{x} \rangle = \sum_{i,j} a_{ij} x_i x_j \geq 0 \) for all \( \overrightarrow{x} = (x_1, x_2, \ldots, x_n) \)

We say that the quadratic form is **positive semi-definite**.

**Lemma** For symmetric matrices \( \mathbf{A} \geq 0 \) & \( \mathbf{B} \geq 0 \implies \text{Trace} (\mathbf{AB}) \geq 0 \).

**Proof:** \( \mathbf{A} \mathbf{B} = \mathbf{A}^{\frac{1}{2}} \mathbf{B}^{\frac{d}{2}} \quad (\mathbf{B} = \mathbf{B}^T) \)

\[ \text{Trace} (\mathbf{AB}) = \text{Trace} (\mathbf{B}^{\frac{d}{2}} \mathbf{A}^{\frac{1}{2}}) \]

\[ \langle \mathbf{B}^{\frac{d}{2}} \mathbf{A}^{\frac{1}{2}} \overrightarrow{x}, \overrightarrow{x} \rangle = \langle \mathbf{A}^{\frac{1}{2}} \overrightarrow{x}, \mathbf{B}^{\frac{d}{2}} \overrightarrow{x} \rangle \]

\[ = \langle \mathbf{A} \overrightarrow{x}, \overrightarrow{x} \rangle \geq 0 \quad \text{where} \quad \overrightarrow{x} = \mathbf{B}^{-\frac{d}{2}} \overrightarrow{x} \]

Thus \( \mathbf{B}^{\frac{d}{2}} \mathbf{A}^{\frac{1}{2}} \geq 0 \) by definition and hence \( \text{Trace} (\mathbf{B}^{\frac{d}{2}} \mathbf{A}^{\frac{1}{2}}) \geq 0 \). (**Remark:** All diagonal elements are \( \geq 0 \) for a pos. semi-definite matrix.)

The trace is their sum.
Assume now that $u \in C^2(\Omega)$.

**Lemma** At an interior maximum point of $u = u(x)$, we have $L(u) \leq 0$; at an interior minimum point $L(u) \geq 0$.

Proof for minimum. By assumption $A \geq 0$ and by the infinitesimal calculus $D \geq 0$ at the minimum point. At the minimum also $\nabla u = 0$. Thus

$$L(u) = \text{trace}(A D) + 0 \geq 0$$

at the minimum point (by previous lemma).

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**Theorem** Assume that $u \in C^2(\Omega)$, $u \in C(\overline{\Omega})$, where $\Omega$ is a bounded domain. If $L(u(x)) \leq 0$ in $\Omega$,

then

$$\min_{x \in \Omega} u(x) = \min_{x \in \Omega_0} u(x)$$

**Proof:** At an interior minimum point the auxiliary function

$$M_\varepsilon(x) = u(x) - \varepsilon x_1$$

satisfies
By Lemma

\[ 0 \leq L(u_\varepsilon) = L(u) + 3a_{11} \varepsilon e^{x_1} - b_1 \varepsilon e^{x_1} \]

\[ \leq 0 - 3e e^{x_1}(\alpha a_{11} + b_1) \]

\[ \leq -3e e^{x_2} (\alpha \inf b_1) \]

\[ \leq -3e \alpha \min(x_2) (\alpha \inf b_1) \]

\[ < 0 \quad \text{if} \quad \alpha > \frac{\inf b_1}{\gamma}, \quad \alpha > 0. \]

This contradiction shows that $u_\varepsilon$ cannot have interior minima. Hence

\[ u(x) - 3e e^{x_2} \geq \min_{x \in \Omega} (u(x) - 3e e^{x_2}) \]

\[ \geq \min_{x \in \partial \Omega} u(x) - 3e \max(x_2) \]

Let $\varepsilon \to 0$. It follows that

\[ u(x) \geq \min_{\partial \Omega} u. \]

The maximum principle for $L(u) \geq 0$ is similar.

Theorem: The solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ to the equation $L(u) = f$ with boundary values $u|_{\partial \Omega} = g$ is unique in the bounded domain $\Omega$ (if it exists).

Proof: $u = u_2 - u_1$, $L(u) = 0$, $u|_{\partial \Omega} = 0$. \qed