First order quasilinear equations

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More general quasilinear equations

Here we look at the general quasilinear equation in two dimensions:

$$au_x + bu_y = c, \tag{1}$$

in which a, b, and c are *really* $a\big(x, y, u(x, y)\big)$, $b\big(x, y, u(x, y)\big)$, and $c\big(x, y, u(x, y)\big)$. (It is the dependence of the coefficients on u that makes the equations *quasi*linear.)

Consider any smooth curve $\big(x(\tau), y(\tau)\big)$. Then, assuming u is a classical solution of (1), we put $z(\tau) = u\big(x(\tau), y(\tau)\big)$, and find

$$z'(\tau) = x'(\tau)u_x + y'(\tau)u_y$$

so that if $x'(\tau) = a$ and $y'(\tau) = b$, then

$$z' = au_x + bu_y = c.$$

This all means that (x, y, z) satify the *characteristic equations*

$$x'(\tau) = a(x, y, z), \qquad y'(\tau) = b(x, y, z), \qquad z'(\tau) = c(x, y, z). \tag{2}$$

To summarize so far: *Assume that u is a classical solution of* (1). *Through each point in the graph of u,*

$$\operatorname{graph}(u) = \{\, (x, y, z) \mid z = u(x, y) \,\},$$

there passes a characteristic curve $\big(x(\tau), y(\tau), z(\tau)\big)$ *solving* (2). *Moreover, each such characteristic curve will lie within the graph of u.*

The solution strategy for (1) can now be explained: Since the graph of a classical solution is a union of characteristic curves, we try to construct solutions by putting together characteristic curves.

The graph of u will be two-dimensional, and a characteristic curve is one-dimensional; so it makes sense to use another variable to keep track of the characteristic curves.

In other words, to construct a solution, we look for three functions $x(\sigma, \tau)$, $y(\sigma, \tau)$, $z(\sigma, \tau)$ which parametrize a characteristic curve as function of τ for each σ. In other words, they should satisfy

$$x_\tau = a, \qquad y_\tau = b, \qquad z_\tau = c$$

where, as always, a, b, and c are considered functions of x, y, and z.[1] Additionally, we assume that these are C^1 functions, and that $\bigl(x(\sigma,\tau), y(\sigma,\tau)\bigr)$ has a C^1 inverse, mapping (x, y) to (σ,τ). Then we can define u by

$$u\bigl(x(\sigma,\tau), y(\sigma,\tau)\bigr) = z(\sigma,\tau).$$

Differentiating this equation with respect to τ yields $x_\tau u_x + y_\tau u_y = z_\tau$, which is the same as (1).

To make this construction more concrete, putting $\tau = 0$ yields a parametric curve γ in the (x, y)-plane: $\bigl(x(\sigma, 0), y(\sigma, 0)\bigr)$, or, if we add a coordinate, a curve Γ in graph(u) parametrized as $\bigl(x(\sigma, 0), y(\sigma, 0), z(\sigma, 0)\bigr)$. The requirement that $\bigl(x(\sigma,\tau), y(\sigma,\tau)\bigr)$ has a C^1 inverse implies that the matrix $\left(\begin{smallmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{smallmatrix}\right)$ is non-singular. At $\tau = 0$ this simply means that Γ is not tangent to the characteristic curves. We refer to this as the *non-characteristic condition.*

We now see what is a natural condition to impose in order to obtain a unique solution to (1): Namely, given a curve γ in the (x, y)-plane, and a function g on γ, assume that the curve Γ given by points (ξ, η, ζ) with (ξ, η) on γ and $\zeta = g(\xi, \eta)$ satisfies the non-characteristic condition. Then (1) has a solution u satisfying the condition

$$u = g \text{ on } \gamma. \tag{3}$$

This solution will exist on a neighbourhood of γ, and be unique there. (Though we must be careful with any end points of γ: They should not be points on γ themselves, as we would then be obliged to extend the solution beyond the end of γ.)

A problem of the form (1) with “initial” data (3) is called a *Cauchy problem* for (1). More generally, a Cauchy problem for a PDE is the problem of solving the PDE along with certain conditions along a curve, or more generally a hypersurface. Often, as is the case here, the PDE more or less dictates the proper form of the Cauchy problem after some analysis.

[1] Well, they were functions of x, y, and u, right? But in this solution, we should think of u and z as being the same. We write u when we emphasize the solution $u(\tau, x)$, but z when we think of the characteristic curves, as in $z(\tau)$ or $z(\sigma,\tau)$. After you have gained some experience, you may find it easier to forget about z and just write u. But this may be too confusing in the beginning.

Transport equations. Here we consider the transport equation with initial data:

$$u_t + au_x = c,$$
$$u(0,x) = u_0(x)$$

where again, in the most general case, we might have $a = a(t,x,u)$ and $c = c(t,x,u)$. The characteristic equations become

$$t' = 1, \qquad x' = a(t,x,z), \qquad z' = c(t,x,z).$$

Since the initial data is given for $t = 0$, we should have $t(0) = 0$, so that $t' = 1$ implies $t = \tau$. Therefore, we forget about τ and use t instead, so what is left of the characteristicc equations is

$$\dot{x} = a(t,x,z), \quad \dot{z} = c(t,x,z).$$

If a depends only on t and x, not on the unknown u, the first equation $\dot{x} = a(t,x)$ can be solved on its own with the initial condition $x(0) = \xi$, and then z equation is solved next. The solution, then, is given by the implicit equation $u(t,x(t)) = z(t)$.

Once you get used to how this all works, it is better to avoid the use of the "z" notation and just write u instead: So while $u(t,x)$ is the solution, $u(t)$ is just the last coordinate on the characteristic curve. The characteristic equations now look like $\dot{x} = a(t,x,u)$ and $\dot{u} = c(t,x,u)$. Here, $\dot{u}$ really becomes the well known *material derivative*, $\dot{u} = Du/Dt$, meaning the derivative of $u(t,x(t))$ along the characteristic path $x(t)$.

A more complicated example. Here we consider the problem

$$-yuu_x + xuu_x = 1, \qquad u(x,0) = x.$$

The characteristic equations with corresponding initial conditions become

$$x' = -yz, \qquad y' = xz, \qquad z' = 1,$$
$$x(0) = \sigma, \qquad y(0) = 0, \qquad z(0) = \sigma.$$

It is easy enough to solve for z: We get $z = \sigma + \tau$. The first two look trickier, until we notice that $xx' + yy' = 0$, so $x^2 + y^2$ is constant. Putting $\tau = 0$ reveals that this constant is σ^2, so we should have

$$x = \sigma \cos\varphi, \quad y = \sigma \sin\varphi$$

with $\varphi = \varphi(\tau)$. The initial conditions are satisfied with $\varphi(0) = 0$, and substituting the proposed x, y into the characteristic equations, we are led to the equation $\varphi' = z =$

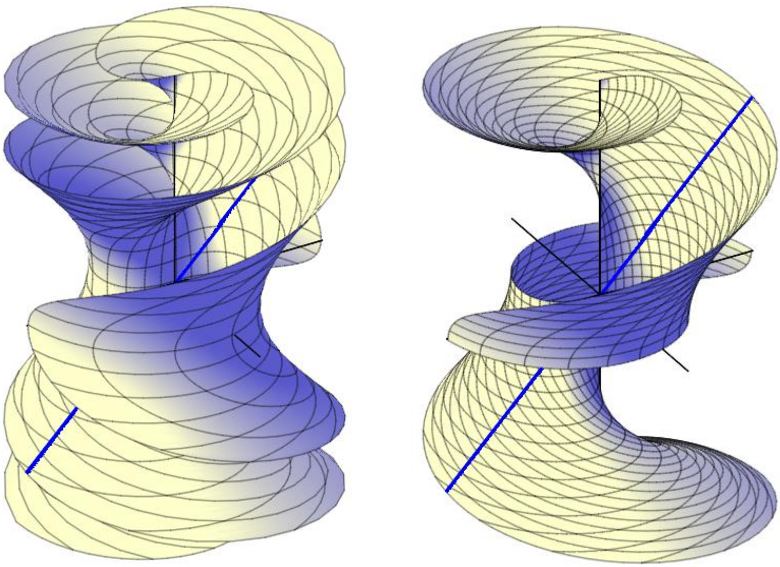
$\sigma + \tau$ (micro-exercise: check that out for yourself). From this we get $\varphi = \sigma\tau + \frac{1}{2}\tau^2$, but as it happens, it is much better to replace τ by $z - \sigma$ everywhere, leading to

$$\varphi = \tfrac{1}{2}(z^2 - \sigma^2).$$

To summarize our findings, we have a parametric surface in $\mathbb{R}^3$ given by

$$\begin{aligned} x &= \sigma\cos\bigl(\tfrac{1}{2}(z^2 - \sigma^2)\bigr), \\ y &= \sigma\sin\bigl(\tfrac{1}{2}(z^2 - \sigma^2)\bigr), \\ z &= z. \end{aligned}$$

Part of the surface is shown on the left below. It is clearly not the graph of a function. However, a suitable neighbourhood of the initial curve $z = x$, $y = 0$, shown as a thick blue line, is the graph of a solution $x = u(x, y)$. As a first step towards determining such a neighbourhood, note the “fold” at $z = 0$, where the graph is vertical. Because of this, we should only consider the two parts of the surface with $z > 0$ and $\sigma > 0$ in one, while $z < 0$ and $\sigma < 0$ in the other. These are shown on the right below.. We still have to restrict the surface further, but shall not try to analyse the possibilities further here.



The general quasilinear equation in higher dimensions

We can do the exact same procedure, with the same arguments, in n dimensions as in the 2-dimensional case. Here we summarize the construction *very* briefly.

A general quasilinear equation then takes the form

$$\boldsymbol{a}\big(\boldsymbol{x}, u(\boldsymbol{x})\big) \cdot \nabla u(\boldsymbol{x}) = c\big(\boldsymbol{x}, u(\boldsymbol{x})\big), \qquad \boldsymbol{x} \in \mathbb{R}^n, \tag{4}$$

with given functions $\boldsymbol{a}$ and c.

The *characteristic equations* become

$$\boldsymbol{x}'(\tau) = \boldsymbol{a}\big(\boldsymbol{x}(\tau), z(\tau)\big), \qquad z'(\tau) = c\big(\boldsymbol{x}(\tau), z(\tau)\big),$$

or written in a more compact form:

$$\boldsymbol{x}' = \boldsymbol{a}(\boldsymbol{x}, z), \qquad z' = c(\boldsymbol{x}, z). \tag{5}$$

Assume now that we are given the PDE (4) with the extra condition

$$u(\boldsymbol{\xi}) = g(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \gamma, \tag{6}$$

where $\gamma \subset \mathbb{R}^n$ is a *hypersurface*, i.e., a surface of dimension $n - 1$. The non-characteristic condition now says that $\boldsymbol{a}\big(\boldsymbol{\xi}, g(\boldsymbol{\xi})\big)$ is not tangent to γ for any $\xi \in \gamma$.

For any $\boldsymbol{\xi} \in \gamma$, standard ODE theory guarantees the existence of a solution of (5) satisfying $\boldsymbol{x}(0) = \boldsymbol{\xi}$ and $z(0) = g(\boldsymbol{\xi})$. Write $\big(\boldsymbol{x}(\tau, \boldsymbol{\xi}), z(\tau, \boldsymbol{\xi})\big)$ for this solution, and define

$$u(\boldsymbol{x}\big(\tau, \boldsymbol{\xi}\big)) = z(\boldsymbol{x}\big(\tau, \boldsymbol{\xi}\big)), \qquad \xi \in \gamma,$$

where again, we can show that this is well defined (for τ sufficiently close to 0) by using the inverse function theorem. Further, the $n + 1$ variables t, $\boldsymbol{\xi}$ are essentially only n variables, because γ is $(n - 1)$-dimensional. The proof that this produces a classical solution is similar to the 2-dimensional case.