Weak maximum principle for the heat equation

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In this note, we consider the standard heat equation

$$u_t - \Delta u = 0 \qquad \text{in } \Omega_T$$

where $\Omega \subset \mathbb{R}^n$ is a *bounded* region, $\Omega_T = (0,T) \times \Omega$ with $T > 0$, and

$$u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big).$$

We think of Ω_T as an open *cylinder* with base Ω and height T. Its closure is a closed cylinder: $\overline{\Omega_T} = [0,T] \times \overline{\Omega}$.

Definition. The *parabolic boundary* of Ω_T is the set

$$\Gamma = \big(\{0\} \times \overline{\Omega}\big) \cup \big([0,T] \times \partial\Omega\big).$$

Clearly, Γ is contained in the normal boundary $\partial\Omega_T$; the difference is

$$\partial\Omega_T \setminus \Gamma = \{T\} \times \Omega.$$

We may call $\{T\} \times \Omega$ the *final boundary* of Ω_T (nonstandard nomenclature).

Observation. If a C^2 function v has a maximum at some point in Ω_T, then $v_t = 0$ and $\Delta v \le 0$ at that point, so we get $v_t - \Delta v \ge 0$ there. Moreover, this holds at the final boundary as well, the only difference being that there, we can only conclude $v_t \ge 0$ and $\Delta v \le 0$. In other words,

$$v_t - \Delta v \ge 0 \quad \text{at any maximum in } \overline{\Omega_T} \setminus \Gamma.$$

We must face a minor technical glitch: The above statement requires that v is C^2 up to and including the final boundary of Ω_T. This complicates the proof of the following theorem, but only a little.

Theorem 1 (The weak maximum principle)**.** *Assume that $u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big)$ satisfies*

$$u_t - \Delta u \le 0.$$

Then $u(t,\boldsymbol{x}) \le \max_\Gamma u$ for all $(t,\boldsymbol{x}) \in \overline{\Omega_T}$. In other words, u achieves its maximum on the parabolic boundary.

Proof. First, to deal with the "minor technical glitch" mentioned above, we shall strengthen the assumptions somewhat, and assume that $u \in C^2\big((0,T] \times \Omega\big)$. We will remove this extra assumption at the end.

Now let $\varepsilon > 0$, and put $v(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) - \varepsilon t$. Then $v_t - \Delta v \le -\varepsilon < 0$, and so it follows *immediately* from the Observation above that v cannot achieve its maximum anywhere other than at Γ. On the other hand, since v is continuous and $\overline{\Omega_T}$ is compact, v does have a maximum in $\overline{\Omega_T}$, and so we must conclude that $v(t,\boldsymbol{x}) \le \max_\Gamma v$ for any $(t,\boldsymbol{x}) \in \overline{\Omega_T}$. But then $u(t,\boldsymbol{x}) = v(t,\boldsymbol{x}) + \varepsilon t \le \max_\Gamma v + \varepsilon T \le \max_\Gamma u + \varepsilon T$. Since this holds for any $\varepsilon > 0$, it finally follows that $u(t,\boldsymbol{x}) \le \max_\Gamma u$, and the proof is complete, with the strengthened assumptions.

We now drop the requirement that $u \in C^2\big((0,T] \times \Omega\big)$. Given any point $(t,\boldsymbol{x}) \in \Omega_T$, pick some T' with $t < T' < T$. Then $u \in C^2\big((0,T'] \times \Omega\big)$, so the first part shows that $u(t,\boldsymbol{x}) \le \max_{\Gamma_{T'}} u$. Here $\Gamma_{T'}$ is tbe parabolic boundary of $\Omega_{T'}$. But $\Gamma_{T'} \subset \Gamma$, so we also have $u(t,\boldsymbol{x}) \le \max_\Gamma u$. Finally, this also holds for $t = T$, since u is continuous on $\overline{\Omega_T}$. This, at last, completes the proof. ∎

It should come as no surprise that there is also a *minimum* principle. It is proved by replacing u by $-u$ in Theorem 1.

Corollary 2 (The weak minimum principle)**.** *Assume that $u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big)$ satisfies*

$$u_t - \Delta u \ge 0.$$

Then $u(t,\boldsymbol{x}) \ge \min_\Gamma u$ for all $(t,\boldsymbol{x}) \in \overline{\Omega_T}$. In other words, u achieves its minimum on the parabolic boundary.

We will mostly be concerned with solutions of the heat equation $u_t - \Delta u = 0$, and for these, both the maximum principle and the minimum principle can be used. But we may also wish to study inhomogeneous equations $u_t - \Delta u = f$, and if f has a definite sign, one or the other principle will apply.

Corollary 3 (Uniqueness for the heat equation)**.** *There exists at most one solution $u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big)$ to the problem*

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega_T, \\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

Here, f and g are given functions on Ω_T and Γ, respectively. (Thus g combines initial values and boundary values in one function.)

Proof. Let u be the difference between two solutions to this problem: Then u solves the same problem, but with $f = 0$ and $g = 0$. Thus u achieves both its minimum and maximum on Γ, but $u = 0$ there, so $u = 0$ everywhere. ∎

The following corollary is proved in essentially the same way, by applying the minimum and maximum principles to $u_1 - u_2$. Note that it immediately implies the preceding corollary by taking $g_1 = g_2$.

Corollary 4 (Continuous dependence on initial data)**.** *Let u_1 and u_2 satisfy*

$$\left.\begin{aligned} \partial_t u_i - \Delta u_i &= f \quad \text{in } \Omega_T, \\ u_i &= g_i \quad \text{on } \Gamma, \end{aligned}\right\} \quad \text{for } i = 1, 2.$$

Then $|u_1 - u_2| \le \max_\Gamma |g_1 - g_2|$.

Unbounded domains: Without further assumptions, the maximum principle is *false* on unbounded domains. This is easiest to see in two dimensions:

Example. With $\Omega = \mathbb{R} \times (0, \pi)$, define $u(t, x, y) = e^x \sin y$ for $t > 0$ and $(x, y) \in \Omega$ (note the lack of time dependence). Then $u_t - \Delta u = 0$ and $u = 0$ on $\partial\Omega$, yet u is not identically zero.

It is possible, but quite a bit harder, to create a similar example with $\Omega = \mathbb{R}$. We describe a famous example by Tikhonov in the appendix at the end of this note. However, with some extra growth condition on the solution, we have the following result:

Theorem 5 (The weak maximum principle on $\mathbb{R}^n$)**.** *Assume that $u \in C\big([0,T] \times \mathbb{R}^n\big) \cap C^2\big((0,T) \times \mathbb{R}^n\big)$ solves $u_t - \Delta u = 0$ in $(0,T) \times \mathbb{R}^n$ with initial data $u(0, \boldsymbol{x}) = g(\boldsymbol{x})$. If $\sup_{\mathbb{R}^n} g = M < \infty$, and if*

$$u(t, \boldsymbol{x}) \le A e^{a|\boldsymbol{x}|^2} \tag{1}$$

for all $(t, \boldsymbol{x})$ and constants A and $a > 0$, then $u(t, \boldsymbol{x}) \le M$ for all $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^n$.

Proof. Inspired by the heat kernel, we define the function B by

$$B(t, \boldsymbol{x}) = t^{-n/2} e^{|\boldsymbol{x}|^2/4t} \qquad \text{for } t > 0 \text{ and } \boldsymbol{x} \in \mathbb{R}^n.$$

A straightforward calculation shows that B satisfies $B_t + \Delta B = 0$ (the *backward heat equation*). Note that B is a strictly decreasing function of t for fixed $\boldsymbol{x}$, and that $B(t, \boldsymbol{x}) \ge t^{-n/2} \to \infty$ when $t \to 0$.

Now let $\varepsilon > 0$ and define

$$v(t, \boldsymbol{x}) = u(t, \boldsymbol{x}) - \varepsilon B(T - t, \boldsymbol{x}).$$

Then $v_t - \Delta v = 0$. We shall apply the maximum principle to the ball $B_R(\boldsymbol{0})$ for some (large) R. Clearly, $v(0, \boldsymbol{x}) < M$ for all $\boldsymbol{x}$. Further, for $|\boldsymbol{x}| = R$ and $0 \le t < T$, we find

$$v(t, \boldsymbol{x}) < A e^{aR^2} - \varepsilon T^{-n/2} e^{R^2/4T} = \big(A e^{(a - 1/4T)R^2} - \varepsilon T^{-n/2}\big) e^{R^2/4T} < M$$

provided $T < 4a$ and R is large enough (depending on ε; and notice that M may be negative). In particular, if R is chosen big enough, $v(t, \boldsymbol{x}) \le M$ whenever $(t, \boldsymbol{x}) \in (0,T) \times \partial B(\boldsymbol{0}, R)$. Therefore, $v(t, \boldsymbol{x}) \le M$ for $(t, \boldsymbol{x}) \in (0,T) \times B(\boldsymbol{0}, R)$. Since R can be as big as we please, this holds for all $\boldsymbol{x} \in \mathbb{R}^n$. Now we get $u(t, \boldsymbol{x}) = v(t, \boldsymbol{x}) + \varepsilon B(T - t, \boldsymbol{x}) \le M + \varepsilon B(T - t, \boldsymbol{x})$. Letting $\varepsilon \to 0$, we conclude that $u(t, \boldsymbol{x}) \le M$.

If $T \ge 4a$, we can use this result repeatedly, first on $[0, T']$, then on $[T', 2T']$ (noting that after the first step we know that $u(T', \boldsymbol{x}) \le M$), and so forth, where $T' < 4a$. ∎

Just as for bounded domains, we can now derive a weak minimum principle, a uniqueness result, and continuous dependence of initial data for the heat equation on $(0,T) \times \mathbb{R}^n$. We just need to add a growth condition like (1) on the solution. The details are left to the reader.

Note that our example above with $\Omega = (0, \pi) \times \mathbb{R}$ satisfies the growth condition (1), showing that we would need a tighter condition to get a maximum principle on that domain.

For the exercises below, we return to bounded domains Ω.

Exercise 1 (Continuous dependence on data, improved)**.** Assume that u_1 and u_2 satisfy

$$\left.\begin{aligned} \partial_t u_i - \Delta u_i &= f_i \quad \text{in } \Omega_T, \\ u_i &= g_i \quad \text{on } \Gamma, \end{aligned}\right\} \quad \text{for } i = 1, 2.$$

Let $\varphi = \sup_{\Omega_T} |f_1 - f_2|$ and $\gamma = \max_\Gamma |g_1 - g_2|$, and show that $|u_1 - u_2| \le \gamma + \varphi T$.

Note that for any t, we can pick $T = t$, so we really get $|u_1 - u_2| \le \gamma + \varphi t$.

Hint: Apply the maximum principle to $u_1 - u_2 - \varphi t$ and $u_2 - u_1 - \varphi t$.

Exercise 2. Show that the maximum (and minimum) principle continues to hold if $u_t - \Delta u$ is replaced by

$$u_t - \Delta u + b(\nabla u),$$

provided the continuous function b satisfies $b(\boldsymbol{0}) = 0$. (For a simple and common example, let $b(\nabla u) = \boldsymbol{b} \cdot \nabla u$.)

Exercise 3. Show that the maximum (and minimum) principle continues to hold if $u_t - \Delta u$ is replaced by the more general

$$u_t - \nabla \cdot (A \nabla u)$$

where the (constant) real $n \times n$ matrix A is symmetric and positive definite.

Here are some ingredients for a proof:

- The *Hessian* of u is defined to be the (symmetric!) $n\times n$ matrix $\mathrm{H}u$ with entries $u_{x_ix_j}$. At an interior maximum point, $\mathrm{H}u$ is negative semidefinite, i.e., $\boldsymbol{y}^T\mathrm{H}\boldsymbol{y} \le 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$. (Short proof: Take the second derivative of $u(\boldsymbol{x} + s\boldsymbol{y})$ with respect to s where $\boldsymbol{x}$ is a maximum point, and put $s = 0$.)

- The *Frobenius inner product* of two real matrices A and B is

$$\langle A, B\rangle_{\mathrm{F}} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ij} = \operatorname{tr}(A^TB).$$

  It turns out that

$$\nabla\cdot(A\nabla u) = \langle A, \mathrm{H}u\rangle_{\mathrm{F}}.$$

- It is known that if A and B are positive semidefinite, then $\langle A, B\rangle_{\mathrm{F}} \ge 0$. (Short proof: Since A is symmetric, we can write $\langle A, B\rangle_{\mathrm{F}} = \operatorname{tr}(AB)$. A will have a positive semidefinite square root $A^{1/2}$. A standard result on the trace gives $\operatorname{tr}(AB) = \operatorname{tr}(A^{1/2}A^{1/2}B) = \operatorname{tr}(A^{1/2}BA^{1/2})$, but $A^{1/2}BA^{1/2}$ is positive semidefinite, and such matrices have nonnegative trace.)

Remark. In many PDE texts, the term $\nabla\cdot(A\nabla u)$ is written out in detail as

$$\nabla\cdot(A\nabla u) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_ix_j}.$$

Pedantically speaking, considering the order in which derivatives are taken, that should be

$$\nabla\cdot(A\nabla u) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_jx_i},$$

but this makes no difference, due to symmetry.

Combining the previous two exercises leads to the maximum/minimum principles for the operator

$$u_t - \nabla\cdot(A\nabla u) + b(\nabla u),$$

but it seemed easier to handle the two extensions separately.

Appendix: Tikhonov's example. Andrey Tikhonov (Андре́й Ти́хонов, 1906–1993) produced an example of a non-zero solution to the heat equation on the real line, with zero initial data. His solution had the form

$$u(t, x) = \sum_{n=0}^{\infty} g_n(t)x^n.$$

Then

$$u_t = \sum_{n=0}^{\infty} g_n'(t)x^n,$$
$$u_{xx} = \sum_{n=0}^{\infty}(n+2)(n+1)g_{n+2}(t)x^n,$$

so we must require

$$g_{n+2}(t) = \frac{g_n'(t)}{(n+2)(n+1)}.$$

Tikhonov chose $g_1(t) = 0$, and hence $g_n(t) = 0$ for all odd n. He also chose $g_0(t) = g(t) = e^{-t^{-a}}$ for some $a > 0$. From the above recurrence we now conclude that

$$u(t, x) = \sum_{k=0}^{\infty} \frac{g^{(2k)}(t)}{(2k)!}x^{2k}.$$

It is apparent that $g^{(2k)}(t)$ equals $e^{-t^{-a}}$ times some linear combination of powers of t. Because $a > 0$, the resulting expression has the limit 0 as $t \to 0$ (from above). Thus also $\lim_{t\to 0} u(t, x) = 0$ for all x.

All of the above requires that the series converge! (Series in plural: That for u itself, and the involved derivatives.) This is the only hard part. But it is *quite* hard; it requires a careful choice of the constant a, and one needs to derive a good upper bound for $|g^{(2k)}(t)|$. We shall not pursue this further.