

Weak maximum principle for the heat equation

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In this note, we consider the standard heat equation

$$u_t - \Delta u = 0 \quad \text{in } \Omega_T$$

where $\Omega \subset \mathbb{R}^n$ is a *bounded* region, $\Omega_T = (0, T) \times \Omega$ with $T > 0$, and

$$u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T).$$

We think of Ω_T as an open *cylinder* with base Ω and height T . Its closure is a closed cylinder: $\overline{\Omega_T} = [0, T] \times \overline{\Omega}$.

Definition. The *parabolic boundary* of Ω_T is the set

$$\Gamma = (\{0\} \times \overline{\Omega}) \cup ([0, T] \times \partial\Omega).$$

Clearly, Γ is contained in the normal boundary $\partial\Omega_T$; the difference is

$$\partial\Omega_T \setminus \Gamma = \{T\} \times \Omega.$$

We may call $\{T\} \times \Omega$ the *final boundary* of Ω_T (nonstandard nomenclature).

Observation. If a C^2 function v has a maximum at some point in Ω_T , then $v_t = 0$ and $\Delta v \leq 0$ at that point, so we get $v_t - \Delta v \geq 0$ there. Moreover, this holds at the final boundary as well, the only difference being that there, we can only conclude $v_t \geq 0$ and $\Delta v \leq 0$. In other words,

$$v_t - \Delta v \geq 0 \quad \text{at any maximum in } \overline{\Omega_T} \setminus \Gamma.$$

We must face a minor technical glitch: The above statement requires that v is C^2 up to and including the final boundary of Ω_T . This complicates the proof of the following theorem, but only a little.

Theorem 1 (The weak maximum principle). *Assume that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies*

$$u_t - \Delta u \leq 0.$$

Then $u(t, \mathbf{x}) \leq \max_{\Gamma} u$ for all $(t, \mathbf{x}) \in \overline{\Omega_T}$. In other words, u achieves its maximum on the parabolic boundary.

v. 2022-09-25

Proof. First, to deal with the “minor technical glitch” mentioned above, we shall strengthen the assumptions somewhat, and assume that $u \in C^2((0, T] \times \Omega)$. We will remove this extra assumption at the end.

Now let $\varepsilon > 0$, and put $v(t, \mathbf{x}) = u(t, \mathbf{x}) - \varepsilon t$. Then $v_t - \Delta v \leq -\varepsilon < 0$, and so it follows *immediately* from the Observation above that v cannot achieve its maximum anywhere other than at Γ . On the other hand, since v is continuous and $\overline{\Omega_T}$ is compact, v does have a maximum in $\overline{\Omega_T}$, and so we must conclude that $v(t, \mathbf{x}) \leq \max_{\Gamma} v$ for any $(t, \mathbf{x}) \in \overline{\Omega_T}$. But then $u(t, \mathbf{x}) = v(t, \mathbf{x}) + \varepsilon t \leq \max_{\Gamma} v + \varepsilon T \leq \max_{\Gamma} u + \varepsilon T$. Since this holds for any $\varepsilon > 0$, it finally follows that $u(t, \mathbf{x}) \leq \max_{\Gamma} u$, and the proof is complete, with the strengthened assumptions.

We now drop the requirement that $u \in C^2((0, T] \times \Omega)$. Given any point $(t, \mathbf{x}) \in \Omega_T$, pick some T' with $t < T' < T$. Then $u \in C^2((0, T'] \times \Omega)$, so the first part shows that $u(t, \mathbf{x}) \leq \max_{\Gamma_{T'}} u$. Here $\Gamma_{T'}$ is the parabolic boundary of $\Omega_{T'}$. But $\Gamma_{T'} \subset \Gamma$, so we also have $u(t, \mathbf{x}) \leq \max_{\Gamma} u$. Finally, this also holds for $t = T$, since u is continuous on $\overline{\Omega_T}$. This, at last, completes the proof. ■

It should come as no surprise that there is also a *minimum* principle. It is proved by replacing u by $-u$ in Theorem 1.

Corollary 2 (The weak minimum principle). *Assume that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies*

$$u_t - \Delta u \geq 0.$$

Then $u(t, \mathbf{x}) \geq \min_{\Gamma} u$ for all $(t, \mathbf{x}) \in \overline{\Omega_T}$. In other words, u achieves its minimum on the parabolic boundary.

We will mostly be concerned with solutions of the heat equation $u_t - \Delta u = 0$, and for these, both the maximum principle and the minimum principle can be used. But we may also wish to study inhomogeneous equations $u_t - \Delta u = f$, and if f has a definite sign, one or the other principle will apply.

Corollary 3 (Uniqueness for the heat equation). *There exists at most one solution $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ to the problem*

$$\begin{aligned} u_t - \Delta u &= f & \text{in } \Omega_T, \\ u &= g & \text{on } \Gamma. \end{aligned}$$

Here, f and g are given functions on Ω_T and Γ , respectively. (Thus g combines initial values and boundary values in one function.)

Proof. Let u be the difference between two solutions to this problem: Then u solves the same problem, but with $f = 0$ and $g = 0$. Thus u achieves both its minimum and maximum on Γ , but $u = 0$ there, so $u = 0$ everywhere. ■

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The following corollary is proved in essentially the same way, by applying the minimum and maximum principles to $u_1 - u_2$. Note that it immediately implies the preceding corollary by taking $g_1 = g_2$.

Corollary 4 (Continuous dependence on initial data). *Let u_1 and u_2 satisfy*

$$\left. \begin{array}{l} \partial_t u_i - \Delta u_i = f \quad \text{in } \Omega_T, \\ u_i = g_i \quad \text{on } \Gamma, \end{array} \right\} \text{ for } i = 1, 2.$$

Then $|u_1 - u_2| \leq \max_{\Gamma} |g_1 - g_2|$.

Unbounded domains: Without further assumptions, the maximum principle is *false* on unbounded domains. In the appendix at the end of this note, we describe a famous example by Tikhonov: A nontrivial solution to the heat equation with $\Omega = \mathbb{R}$ with zero initial value. However, with some extra growth condition on the solution, we have the following result:

Theorem 5 (The weak maximum principle on \mathbb{R}^n). *Assume that $u \in C([0, T] \times \mathbb{R}^n) \cap C^2((0, T) \times \mathbb{R}^n)$ solves $u_t - \Delta u = 0$ in $(0, T) \times \mathbb{R}^n$ with initial data $u(0, \mathbf{x}) = g(\mathbf{x})$. If $\sup_{\mathbb{R}^n} g = M < \infty$, and if*

$$u(t, \mathbf{x}) \leq Ae^{a|\mathbf{x}|^2} \quad (1)$$

for all (t, \mathbf{x}) , where A and $a > 0$ are constants, then $u(t, \mathbf{x}) \leq M$ for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n$.

Proof. Inspired by the heat kernel, we define the function B by

$$B(t, \mathbf{x}) = t^{-n/2} e^{-|\mathbf{x}|^2/4t} \quad \text{for } t > 0 \text{ and } \mathbf{x} \in \mathbb{R}^n.$$

A straightforward calculation shows that B satisfies $B_t + \Delta B = 0$ (the *backward heat equation*). Note that B is a strictly decreasing function of t for fixed \mathbf{x} , and that $B(t, \mathbf{x}) \geq t^{-n/2} \rightarrow \infty$ when $t \rightarrow 0$.

Now let $\varepsilon > 0$ and define

$$v(t, \mathbf{x}) = u(t, \mathbf{x}) - \varepsilon B(T - t, \mathbf{x}).$$

Then $v_t - \Delta v = 0$. We shall apply the maximum principle to the ball $B_R(\mathbf{0})$ for some (large) R . Clearly, $v(0, \mathbf{x}) < M$ for all \mathbf{x} . Further, for $|\mathbf{x}| = R$ and $0 \leq t < T$, we find

$$v(t, \mathbf{x}) < Ae^{aR^2} - \varepsilon T^{-n/2} e^{R^2/4T} = (Ae^{(a-1/4T)R^2} - \varepsilon T^{-n/2}) e^{R^2/4T} < M$$

provided $4aT < 1$ and R is large enough (depending on ε ; and notice that M may be negative). In particular, if R is chosen big enough, $v(t, \mathbf{x}) \leq M$ whenever $(t, \mathbf{x}) \in (0, T) \times \partial B(\mathbf{0}, R)$. Therefore, $v(t, \mathbf{x}) \leq M$ for $(t, \mathbf{x}) \in (0, T) \times B(\mathbf{0}, R)$. Since R can be as

big as we please, this holds for all $\mathbf{x} \in \mathbb{R}^n$. Now we get $u(t, \mathbf{x}) = v(t, \mathbf{x}) + \varepsilon B(T - t, \mathbf{x}) \leq M + \varepsilon B(T - t, \mathbf{x})$. Letting $\varepsilon \rightarrow 0$, we conclude that $u(t, \mathbf{x}) \leq M$.

If $4aT \geq 1$, we can use this result repeatedly, first on $[0, T']$, then on $[T', 2T']$ (noting that after the first step we know that $u(T', \mathbf{x}) \leq M$), and so forth, where $4aT' < 1$. ■

Just as for bounded domains, we can now derive a weak minimum principle, a uniqueness result, and continuous dependence of initial data for the heat equation on $(0, T) \times \mathbb{R}^n$. We just need to add a growth condition like (1) on the solution. The details are left to the reader.

Note that our example above with $\Omega = (0, \pi) \times \mathbb{R}$ satisfies the growth condition (1), showing that we would need a tighter condition to get a maximum principle on that domain.

For the exercises below, we return to bounded domains Ω .

Exercise 1 (Continuous dependence on data, improved). Assume that u_1 and u_2 satisfy

$$\left. \begin{array}{l} \partial_t u_i - \Delta u_i = f_i \quad \text{in } \Omega_T, \\ u_i = g_i \quad \text{on } \Gamma, \end{array} \right\} \text{ for } i = 1, 2.$$

Let $\varphi = \sup_{\Omega_T} |f_1 - f_2|$ and $\gamma = \max_{\Gamma} |g_1 - g_2|$, and show that $|u_1 - u_2| \leq \gamma + \varphi T$.

Note that for any t , we can pick $T = t$, so we really get $|u_1 - u_2| \leq \gamma + \varphi t$.

Hint: Apply the maximum principle to $u_1 - u_2 - \varphi t$ and $u_2 - u_1 - \varphi t$.

Exercise 2. Show that the maximum (and minimum) principle continues to hold if $u_t - \Delta u$ is replaced by

$$u_t - \Delta u + b(\nabla u),$$

provided the continuous function b satisfies $b(\mathbf{0}) = 0$. (For a simple and common example, let $b(\nabla u) = \mathbf{b} \cdot \nabla u$.)

Exercise 3. Show that the maximum (and minimum) principle continues to hold if $u_t - \Delta u$ is replaced by the more general

$$u_t - \nabla \cdot (A \nabla u)$$

where the (constant) real $n \times n$ matrix A is symmetric and positive definite.

Here are some ingredients for a proof:

- The *Hessian* of u is defined to be the (symmetric!) $n \times n$ matrix Hu with entries $u_{x_i x_j}$. At an interior maximum point, Hu is negative semidefinite, i.e., $\mathbf{y}^T H \mathbf{y} \leq 0$ for all $\mathbf{y} \in \mathbb{R}^n$. (Short proof: Take the second derivative of $u(\mathbf{x} + s\mathbf{y})$ with respect to s where \mathbf{x} is a maximum point, and put $s = 0$.)

- The *Frobenius inner product* of two real matrices A and B is

$$\langle A, B \rangle_{\mathbb{F}} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} = \text{tr}(A^T B).$$

It turns out that

$$\nabla \cdot (A \nabla u) = \langle A, H u \rangle_{\mathbb{F}}.$$

- It is known that if A and B are positive semidefinite, then $\langle A, B \rangle_{\mathbb{F}} \geq 0$. (Short proof: Since A is symmetric, we can write $\langle A, B \rangle_{\mathbb{F}} = \text{tr}(AB)$. A will have a positive semidefinite square root $A^{1/2}$. A standard result on the trace gives $\text{tr}(AB) = \text{tr}(A^{1/2} A^{1/2} B) = \text{tr}(A^{1/2} B A^{1/2})$, but $A^{1/2} B A^{1/2}$ is positive semidefinite, and such matrices have nonnegative trace.)

Remark. In many PDE texts, the term $\nabla \cdot (A \nabla u)$ is written out in detail as

$$\nabla \cdot (A \nabla u) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} u_{x_i x_j}.$$

Pedantically speaking, considering the order in which derivatives are taken, that should be

$$\nabla \cdot (A \nabla u) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} u_{x_j x_i},$$

but this makes no difference, due to symmetry.

Combining the previous two exercises leads to the maximum/minimum principles for the operator

$$u_t - \nabla \cdot (A \nabla u) + b(\nabla u),$$

but it seemed easier to handle the two extensions separately.

Appendix: Tikhonov's example. Andrey Tikhonov (Андрей Тихонов, 1906–1993) produced an example of a non-zero solution to the heat equation on the real line, with zero initial data. His solution had the form

$$u(t, x) = \sum_{n=0}^{\infty} g_n(t) x^n.$$

Then

$$\begin{aligned} u_t &= \sum_{n=0}^{\infty} g'_n(t) x^n, \\ u_{xx} &= \sum_{n=0}^{\infty} (n+2)(n+1) g_{n+2}(t) x^n, \end{aligned}$$

so we must require

$$g_{n+2}(t) = \frac{g'_n(t)}{(n+2)(n+1)}.$$

Tikhonov chose $g_1(t) = 0$, and hence $g_n(t) = 0$ for all odd n . He also chose $g_0(t) = g(t) = e^{-t^a}$ for some $a > 0$. From the above recurrence we now conclude that

$$u(t, x) = \sum_{k=0}^{\infty} \frac{g^{(2k)}(t)}{(2k)!} x^{2k}.$$

It is apparent that $g^{(2k)}(t)$ equals e^{-t^a} times some linear combination of powers of t . Because $a > 0$, the resulting expression has the limit 0 as $t \rightarrow 0$ (from above). Thus also $\lim_{t \rightarrow 0} u(t, x) = 0$ for all x .

All of the above requires that the series converge! (Series in plural: That for u itself, and the involved derivatives.) This is the only hard part. But it is *quite* hard; it requires a careful choice of the constant a , and one needs to derive a good upper bound for $|g^{(2k)}(t)|$. We shall not pursue this further.