

Harmonicfunctionology

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The Laplace operator on \mathbb{R}^n

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

plays a rôle in the wave and heat equations, but even more fundamentally, in the *Laplace equation*

$$\Delta u = 0 \quad (1)$$

and the *Poisson equation* $-\Delta u = f$ where f is a given function.

A C^2 solution of (1) is called *harmonic*. (Later, in Theorem 4, we will find that harmonic functions are in fact infinitely differentiable.)

Much is gained from the trivial observation that $\Delta u = \nabla \cdot \nabla u$ together with various applications of the divergence theorem or two of its corollaries, Green's first and second identities.

Let's get started by simply integrating Δu over a bounded domain ω with piecewise C^1 boundary:¹

$$\int_{\omega} \Delta u(\mathbf{x}) \, d^n \mathbf{x} = \int_{\omega} \nabla \cdot \nabla u(\mathbf{x}) \, d^n \mathbf{x} = \int_{\partial \omega} \partial_n u(\mathbf{x}) \, dS(\mathbf{x}). \quad (2)$$

This immediately proves

Proposition 1. *If a C^2 function u on a domain Ω is harmonic, then*

$$\int_{\partial \omega} \partial_n u \, dS = 0 \quad (3)$$

for all bounded domains ω with $\bar{\omega} \subset \Omega$ having piecewise C^1 boundary.

Conversely, if (3) holds for every ball $\omega = B_r(\mathbf{x})$ whose closure lies within Ω , then u is harmonic.

Proof. We have already proved the first part. For the converse, (3) and (2) imply that the average of Δu over any ball is zero. By letting the radius of the ball $B_r(\mathbf{x})$ tend to zero, we conclude that $\Delta u(\mathbf{x}) = 0$. ■

Definition. The (*radius r*) *spherical average* of a function u at a point \mathbf{x} is defined to be

$$\bar{u}_{\mathbf{x}}(r) = \int_{\partial B_r(\mathbf{x})} u \, dS = \int_{\mathbb{S}^{n-1}} u(\mathbf{x} + r\mathbf{y}) \, dS(\mathbf{y}),$$

¹Notation used here and elsewhere is explained at the end of this note.

where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere and the “barred” integral signs denote the average:

$$\int_{\partial B_r(\mathbf{x})} u \, dS = \frac{1}{A_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u \, dS,$$

and A_n is the area of \mathbb{S}^{n-1} . Note that the second integral in the definition of $\bar{u}_{\mathbf{x}}(r)$ makes sense even for $r < 0$; thus, we adopt this as the definition for all real r for which the integrand is defined on \mathbb{S}^{n-1} . We see that $\bar{u}_{\mathbf{x}}$ is an *even* function; it is C^k if u is C^k , and $\bar{u}_{\mathbf{x}}(0) = u(\mathbf{x})$.

When ω is a ball, we can rewrite (2) in terms of spherical averages: Noting that the volume of the ball $B_r(\mathbf{x})$ is $A_n r^n / n$, we find

$$\int_{B_r(\mathbf{x})} \Delta u(\mathbf{y}) \, d^n \mathbf{y} = \frac{n}{r} \int_{\partial B_r(\mathbf{x})} \partial_n u(\mathbf{y}) \, dS(\mathbf{y}) = \frac{n}{r} \int_{\mathbb{S}^{n-1}} \partial_r u(\mathbf{x} + r\mathbf{y}) \, dS(\mathbf{y}),$$

where we can move the r derivative outside the integral, and arrive at

$$\int_{B_r(\mathbf{x})} \Delta u(\mathbf{y}) \, d^n \mathbf{y} = \frac{n}{r} \bar{u}'_{\mathbf{x}}(r). \quad (4)$$

Along with $\bar{u}_{\mathbf{x}}(0) = u(\mathbf{x})$, this implies

Theorem 2 (The mean value property of harmonic functions). *A C^2 function u on a domain Ω is harmonic if and only if $\bar{u}_{\mathbf{x}}(r) = u(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ and all r for which $\bar{B}_{|r|}(\mathbf{x}) \subset \Omega$.*

In general, we say a function u satisfies the *mean value property* if $\bar{u}_{\mathbf{x}}(r) = u(\mathbf{x})$ whenever $\bar{B}_{|r|}(\mathbf{x}) \subset \Omega$. We shall see below (Theorem 4) that the mean value property characterizes harmonic functions. But first, we collect an easy consequence of (4).²

Proposition 3. *For any C^2 function u , we have*

$$\Delta u(\mathbf{x}) = n \bar{u}''_{\mathbf{x}}(0).$$

Proof. The function $\bar{u}_{\mathbf{x}}$ is even, so $\bar{u}'_{\mathbf{x}}(0) = 0$. Therefore, letting $r \rightarrow 0$ in (4), we arrive at the stated result. ■

Theorem 4 (The mean value property and regularity). *Assume that a continuous function u satisfies the mean value property on a domain Ω . Then u is infinitely differentiable, and is therefore harmonic. In particular, every harmonic function is infinitely differentiable.*

²Faded out because although the result is interesting, we shall not use it.

Proof. This proof may seem long, but only because we use it to develop some tools that have wider applicability.

First, define a *standard mollifier* $\rho: \mathbb{R}^n \rightarrow \mathbb{R}$. Here is one of many possible definitions:

$$\rho(\mathbf{x}) = \begin{cases} ae^{1/(|\mathbf{x}|^2-1)}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \geq 1, \end{cases}$$

where the constant $a > 0$ is chosen to ensure that

$$\int_{\mathbb{R}^n} \rho \, d^n \mathbf{x} = 1.$$

That is one of the defining qualities of a standard mollifier. The others are: That $\rho \geq 0$ everywhere, that it vanishes outside the unit ball, that it is infinitely differentiable, and is radially symmetric – that is, a function of $|\mathbf{x}|$ alone.

For any $\delta > 0$ we can squeeze the mollifier to fit inside a ball of radius δ :

$$\rho_\delta(\mathbf{x}) = \frac{1}{\delta^n} \rho\left(\frac{\mathbf{x}}{\delta}\right),$$

so that ρ_δ also has integral 1, but vanishes outside the ball $B_\delta(0)$.

Now we consider the convolution product

$$u * \rho_\delta(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{y}) \rho_\delta(\mathbf{x} - \mathbf{y}) \, d^n \mathbf{y}.$$

This is defined for all $\mathbf{x} \in \Omega$ with a distance less than δ to the complement of Ω . Thus, for any $\mathbf{x} \in \Omega$, we can make δ small enough so that $u * \rho_\delta$ is defined at \mathbf{x} .

Moreover, $u * \rho_\delta$ is infinitely differentiable: This is proved by differentiating with respect to the components of \mathbf{x} under the integral sign, as much as you like.

Finally, the mean value property of u and the radial symmetry of ρ_δ combine to ensure that $u(\mathbf{x}) = u * \rho(\mathbf{x})$ for all \mathbf{x} where $u * \rho$ is defined.

For a detailed argument, write

$$u * \rho_\delta(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{x} - \mathbf{y}) \rho_\delta(\mathbf{y}) \, d^n \mathbf{y}$$

and write the integral in polar form:

$$\begin{aligned} u * \rho_\delta(\mathbf{x}) &= \int_0^\delta \int_{\partial B_r(0)} u(\mathbf{x} - \mathbf{y}) \rho_\delta(\mathbf{y}) \, dS(\mathbf{y}) \, dr \\ &= \int_0^\delta \int_{\mathbb{S}^{n-1}} u(\mathbf{x} - r\mathbf{y}) \rho_\delta(r\mathbf{y}) \, dS(\mathbf{y}) r^{n-1} \, dr. \end{aligned} \quad (5)$$

Now use the radial symmetry: $\bar{\rho}_\delta(r) := \rho_\delta(r\mathbf{y})$ is independent of $\mathbf{y} \in \mathbb{S}^{n-1}$, so this factor can be moved outside the inner integral. Next, use the mean value property of u :

$$\int_{\mathbb{S}^{n-1}} u(\mathbf{x} - r\mathbf{y}) \rho_\delta(r\mathbf{y}) \, dS(\mathbf{y}) = A_n u(\mathbf{x}) \bar{\rho}_\delta(r).$$

But $u(\mathbf{x})$ is a constant, which we move outside the outer integral in (5). We are left with

$$u * \rho_\delta(\mathbf{x}) = u(\mathbf{x}) \int_0^\delta r^{n-1} A_n \bar{\rho}_\delta(r) \, dr = u(\mathbf{x}) \int_{B_\delta(0)} \rho_\delta(\mathbf{z}) \, d^n \mathbf{z} = u(\mathbf{x}).$$

Since $u * \rho_\delta$ is C^∞ , then so is u , wherever $u * \rho_\delta$ is defined. By making δ as small as we wish, we conclude that $u \in C^\infty(\Omega)$, as was our goal. ■

Corollary 5. *A uniform limit of a sequence of harmonic functions is harmonic.*

Proof. Left as an exercise for the reader. ■

The gradient bound below will turn out to be useful later on. But first, note that the same method used to prove that $u(\mathbf{x}) = u * \rho_\delta(\mathbf{x})$ above also shows that $u(\mathbf{x})$ is the average of u over a *ball* centred at \mathbf{x} .

Proposition 6 (Gradient bound). *Assume that u is a bounded harmonic function on a domain $\Omega \subset \mathbb{R}^n$. Then*

$$|\nabla u(\mathbf{x})| \leq \frac{n \sup_\Omega |u|}{\text{dist}(\mathbf{x}, \partial\Omega)}$$

Proof. Let \mathbf{v} be any unit vector. Each component of ∇u is harmonic, and hence so is $\mathbf{v} \cdot \nabla u$. Applying the mean value theorem (ball version) where $\bar{B}_r(\mathbf{x}) \subset \Omega$, we get

$$\begin{aligned} \mathbf{v} \cdot \nabla u(\mathbf{x}) &= \frac{n}{A_n r^n} \int_{B_r(\mathbf{x})} \mathbf{v} \cdot \nabla u(\mathbf{y}) \, d^n \mathbf{y} \\ &= \frac{n}{A_n r^n} \int_{\partial B_r(\mathbf{x})} \mathbf{n} \cdot \mathbf{v} u(\mathbf{y}) \, dS(\mathbf{y}) \quad \text{because } \mathbf{v} \cdot \nabla u = \nabla \cdot (\mathbf{v} u), \end{aligned}$$

and therefore

$$|\mathbf{v} \cdot \nabla u(\mathbf{x})| \leq \frac{n}{A_n r^n} A_n r^{n-1} \sup_\Omega |u| = \frac{n \sup_\Omega |u|}{r}.$$

Now let $r \rightarrow \text{dist}(\mathbf{x}, \partial\Omega)$ to finish the proof. ■

The maximum principle

Definition. A C^2 function u is called *subharmonic* if $\Delta u \geq 0$, and *superharmonic* if $\Delta u \leq 0$. Thus it is harmonic if and only if it is both subharmonic and superharmonic. (The reason for the naming will become clear later; see Corollary 9.) Clearly, u is superharmonic if and only if $-u$ is subharmonic.

Note that, in the one-dimensional case ($n = 1$), subharmonic functions are *convex*, while superharmonic functions are *concave*.³ This is a great help to intuition – think of this special case as we explore the properties of sub- and superharmonic functions below!

Theorem 7 (Strong maximum principle). *Assume that $u \in C^2(\Omega)$ is subharmonic in a region $\Omega \subseteq \mathbb{R}^n$. If u has a global maximum in Ω , then u is constant.*

Proof. Let M be the global maximum of u , and put

$$S = \{x \in \Omega \mid u(x) = M\}.$$

Then S is a closed subset of Ω , by the continuity of u . It is also nonempty by assumption.

Consider any $x \in S$. From (4) and the subharmonicity of u , we get $\tilde{u}'_x(r) \geq 0$ for $r > 0$. Thus we get $\tilde{u}_x(r) \geq \tilde{u}_x(0) = u(x) = M$ for $r > 0$ (so long as $\bar{B}(x, r) \subset \Omega$). But $u \leq M$ everywhere, and if $u < M$ anywhere on the sphere $\partial B_r(x)$, we would get $\tilde{u}_x(r) < M$. Thus $u = M$ in some neighbourhood of x . This means that S is open.

We have shown that S is an open, closed, and nonempty subset of the connected set Ω . Therefore $S = \Omega$, and the proof is complete. ■

Remark. Obviously, we obtain a strong *minimum* principle for *superharmonic* functions by multiplying by -1 . In particular, a non-constant harmonic function cannot attain a minimum or maximum value anywhere in Ω .

Corollary 8 (Weak maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is subharmonic. Then*

$$\max \{u(x) \mid x \in \bar{\Omega}\} = \max \{u(x) \mid x \in \partial\Omega\}.$$

In particular, a harmonic function which is continuous on $\bar{\Omega}$ attains its minimum and maximum values on the boundary $\partial\Omega$.

Proof. The weak principle is an obvious consequence of the strong principle together with the existence of a maximum of the continuous function u on the compact set $\bar{\Omega}$.

³Or, as calculus textbooks perversely(?) call it, “concave up” and “concave down”, respectively.

However, it is worth noting that a much more elementary proof exists. Namely, for any $\varepsilon > 0$, let $v(x) = u(x) + \varepsilon|x|^2$, and note that then $\Delta v > 0$. But $\Delta v(x) \leq 0$ if x is an interior maximum point for v , so v cannot have any maximum in the interior. Thus for any $x \in \Omega$,

$$u(x) = v(x) - \varepsilon|x|^2 \leq \max_{\partial\Omega} v \leq \max_{\partial\Omega} u + \varepsilon \max_{x \in \partial\Omega} |x|^2.$$

Now let $\varepsilon \rightarrow 0$ to arrive at the conclusion $u(x) \leq \max_{\partial\Omega} u$. ■

Our next result explains the terms *sub-* and *superharmonic*: A subharmonic function is below, and a superharmonic above, a harmonic function given the same boundary data.

Corollary 9. *Assume that Ω is a bounded domain, and that $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$, with u harmonic in Ω . If v is subharmonic and $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω , while if v is superharmonic and $v \geq u$ on $\partial\Omega$, then $v \geq u$ in Ω .*

Proof. Apply the weak maximum principle to $v - u$ if v is subharmonic, or to $u - v$ if v is superharmonic. ■

Remark. Corollary 9 suggests a strategy for proving existence of a solution to the Dirichlet problem for the Laplace equation: Assume we are trying to find a harmonic function on Ω with the given boundary value g . Consider the pointwise *supremum* of all subharmonic functions which are $\leq g$ on $\partial\Omega$, and the pointwise *infimum* of all superharmonic functions which are $\geq g$ on $\partial\Omega$. If the two functions coincide, they might provide a solution to the problem. (The truth isn’t quite that rosy, depending on the domain.) This is the basis for *Perron’s method*, described later.

The Poisson equation

We now turn our study to the *Poisson equation*:

$$-\Delta u = f \tag{6}$$

where f is a known continuous function. (It *must* be continuous to allow for classical, i.e., C^2 , solutions u .)

Referring all the way back to (2), we quickly get the following generalization of Proposition 1:

Proposition 10. *A C^2 function u on a domain Ω solves the Poisson equation (6) if and only if*

$$-\int_{\partial\omega} \partial_n u(\mathbf{x}) \, dS(\mathbf{x}) = \int_{\omega} f(\mathbf{x}) \, d^n \mathbf{x} \tag{7}$$

for all bounded domains ω with $\bar{\omega} \subset \Omega$ having piecewise C^1 boundary. It is sufficient to consider balls $\omega = B_r(\mathbf{x})$.

As an example, we consider a Poisson equation with a radially symmetric right hand side $f(\mathbf{x}) = \bar{f}(|\mathbf{x}|)$. We expect to find a radially symmetric solution $u(\mathbf{x}) = \bar{u}(|\mathbf{x}|)$. Now (7) with $\omega = B_r(\mathbf{0})$ becomes

$$-A_n r^{n-1} \bar{u}'(r) = A_n \int_0^r \bar{f}(s) s^{n-1} \, ds.$$

Taking the derivative and rearranging turns this into the ODE

$$-\frac{1}{r^{n-1}} \frac{d}{dr} (r^{n-1} \bar{u}'(r)) = \bar{f}(r). \tag{8}$$

A direct calculation reveals that indeed,

$$\Delta \bar{u}(|\mathbf{x}|) = \bar{u}''(|\mathbf{x}|) + \frac{n-1}{|\mathbf{x}|} \bar{u}'(|\mathbf{x}|) = \frac{1}{r^{n-1}} \frac{d}{dr} (r^{n-1} \bar{u}'(r)) \Big|_{r=|\mathbf{x}|}$$

so a solution to (8) will in fact solve the Poisson equation in the radially symmetric case.

The ODE (8) is easily integrated:

$$-r^{n-1} \bar{u}'(r) = \int_0^r \bar{f}(s) s^{n-1} \, ds = \frac{1}{A_n} \int_{B_r(\mathbf{0})} f(\mathbf{x}) \, d^n \mathbf{x} =: \frac{m(r)}{A_n}.$$

If we let f approach a Dirac's δ , we would end up with $m(r) = 1$ for all r . Thus, after integrating once more, we arrive at the solution $u = \Phi$, where $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{-\ln(|\mathbf{x}|)}{2\pi} && \text{for } n = 2, \\ \Phi(\mathbf{x}) &= \frac{1}{(n-2)A_n |\mathbf{x}|^{n-2}} && \text{for } n \geq 3. \end{aligned}$$

For $n \geq 3$, this gives the unique radially symmetric solution which vanishes at infinity; for $n = 2$, however, no particular value of the dropped constant of integration distinguishes itself.

Now, put $m = 1$, replace $f(\mathbf{x})$ by $\varepsilon^{-n} f(\mathbf{x}/\varepsilon)$, and let $\varepsilon \rightarrow 0$. The resulting solution u will converge pointwise to Φ (except at $\mathbf{x} = \mathbf{0}$), while f becomes a Dirac δ in the limit. Thus we are tempted to conclude that

$$-\Delta \Phi = \delta$$

This is indeed true, but we first need to get into the theory of distributions in order to understand the rigorous meaning of the above equation.

However, we can make one useful observation. The following lemma says precisely what you would expect, given Proposition 10 and $-\Delta \Phi = \delta$:

Lemma 11. *Given a bounded region $\omega \subset \mathbb{R}^n$ with piecewise C^1 boundary and $\mathbf{0} \notin \partial\omega$,*

$$\int_{\partial\omega} \partial_n \Phi \, dS = \begin{cases} -1 & \text{if } \mathbf{0} \in \omega, \\ 0 & \text{if } \mathbf{0} \notin \bar{\omega}. \end{cases}$$

Proof. The second case is a consequence of the harmonicity of Φ and Proposition 1. For the first case, it is easily verified by direct calculation in the case of a ball $\omega = B_r(\mathbf{0})$. In the general case for $\mathbf{0} \in \omega$, pick $r > 0$ small enough so that $\bar{B}_r(\mathbf{0}) \subset \omega$, and apply the second case to $\omega \setminus B_r(\mathbf{0})$. Thus

$$0 = \int_{\partial(\omega \setminus B_r(\mathbf{0}))} \partial_n \Phi \, dS = \int_{\partial\omega} \partial_n \Phi \, dS - \int_{\partial B_r(\mathbf{0})} \partial_n \Phi \, dS = \int_{\partial\omega} \partial_n \Phi \, dS + 1,$$

and the proof is complete. (The minus sign in the above calculation is because the unit normal pointing *out* of $\omega \setminus B_r(\mathbf{0})$ at $\partial B_r(\mathbf{0})$ points *into* $B_r(\mathbf{0})$.) ■

We call Φ the *fundamental solution* for the Poisson equation. Even lacking the abstract theory, we can use it to solve the general Poisson equation.

Proposed solution to the Poisson equation:

$$u(\mathbf{x}) = \Phi * f(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{y})f(\mathbf{x} - \mathbf{y}) d^n \mathbf{y} \quad (9)$$

We expect this to solve the equation because formally, we get

$$-\Delta u = -\Delta(\Phi * f) = (-\Delta\Phi) * f = \delta * f = f.$$

However, this calculation is hard to justify by elementary means, but we can do it with a bit of help from Proposition 10.

Theorem 12. Assume that $f \in C_c^2(\mathbb{R}^n)$. Then the function u given by (9) is a solution to the Poisson equation (6).

Proof. First, the assumptions imply that $u \in C^2$, since Φ is locally integrable. Let $\omega \subset \mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary. Then we calculate:

$$\begin{aligned} -\int_{\partial\omega} \partial_n u(\mathbf{x}) dS(\mathbf{x}) &= -\int_{\partial\omega} \partial_n \left(\int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y})f(\mathbf{y}) d^n \mathbf{y} \right) dS(\mathbf{x}) \\ &= -\int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\partial\omega} \partial_n \Phi(\mathbf{x} - \mathbf{y}) dS(\mathbf{x}) \right) d^n \mathbf{y} \\ &= -\int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\partial(\omega - \mathbf{y})} \partial_n \Phi(\mathbf{z}) dS(\mathbf{z}) \right) d^n \mathbf{y} \\ &\stackrel{*}{=} \int_{\mathbb{R}^n} f(\mathbf{y}) [\mathbf{0} \in \omega - \mathbf{y}] d^n \mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{y}) [\mathbf{y} \in \omega] d^n \mathbf{y} \\ &= \int_{\omega} f(\mathbf{y}) d^n \mathbf{y}. \end{aligned}$$

In the above calculation, $\omega - \mathbf{y} = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x} \in \omega\}$, and the square brackets are *Iverson brackets*: For any statement S ,

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true,} \\ 0 & \text{if } S \text{ is false.} \end{cases}$$

We used Lemma 11 in the equality marked with an asterisk. By Proposition 10, the proof is now complete. ■

A more conventional proof of Theorem 12

Our main tool is Green's second identity. We often need to employ it on a region containing the singularity of Φ , which is of course not allowed. So the trick is to remove a ball centered on the singularity, use Green's second identity on the rest of the region, and then let the radius of the ball tend to zero. This leads to two limits handled by the following two lemmas.

Lemma 13. If the function v is continuous near $\mathbf{0}$, then

$$-\int_{\partial B_r(\mathbf{0})} v \partial_n \Phi dS = \int_{\partial B_r(\mathbf{0})} v dS \rightarrow v(\mathbf{0}) \quad \text{as } r \rightarrow 0.$$

Proof. The identity follows from the fact that the normal derivative on $\partial B_r(\mathbf{0})$ is the radial derivative, with the constant value $\partial_n \Phi = -1/(A_n r^{n-1})$ on the sphere. And then the limit follows by continuity of v . ■

Lemma 14. If the function v is C^1 near $\mathbf{0}$, then

$$\int_{\partial B_r(\mathbf{0})} \Phi \partial_n v dS \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Proof. The normal derivative $\partial_n v$ is really $\mathbf{n} \cdot \nabla v$, but \mathbf{n} is a unit vector, and ∇v is bounded near $\mathbf{0}$, since v is assumed to be C^1 . Because the integral of $|\Phi|$ tends to zero as $r \rightarrow 0$, the integral of the first term will vanish in the limit. ■

Theorem 15. Assume that $f \in C_c^2(\mathbb{R}^n)$. Then the function u given by (9) is a solution to the Poisson equation (6).

Proof. Thanks to translation invariance, we only need to prove that $-\Delta(\Phi * f)(\mathbf{0}) = f(\mathbf{0})$. (For any $\mathbf{x}_0 \in \mathbb{R}^n$, put $\tilde{f}(\mathbf{x}) = f(\mathbf{x}_0 + \mathbf{x})$. Then $\Phi * \tilde{f}(\mathbf{x}) = \Phi * f(\mathbf{x}_0 + \mathbf{x})$, so $-\Delta(\Phi * \tilde{f})(\mathbf{0}) = \tilde{f}(\mathbf{0})$ implies $-\Delta(\Phi * f)(\mathbf{x}) = f(\mathbf{x})$.)

First, we note that

$$\int_{\partial B_r(\mathbf{0})} \Phi d^n \mathbf{x} = \begin{cases} -r \ln r & \text{for } n = 2, \\ r/(n-2) & \text{for } n \geq 3. \end{cases}$$

Integrating this with respect to r , we conclude that Φ is integrable (meaning the integral of $|\Phi|$ is finite) over $B_r(\mathbf{0})$, and hence over any bounded subset of \mathbb{R}^n . From this, we conclude that not only is

$$u(\mathbf{x}) = \Phi * f(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{y})f(\mathbf{x} - \mathbf{y}) d^n \mathbf{y}$$

well defined, but u is C^2 as well, and in fact

$$\Delta u = \Phi * \Delta f.$$

This is easy if f has compact support; it is also true if f and its derivatives up to second order vanish sufficiently fast at infinity, but we are not going to bother with this refinement.

In particular,

$$\Delta u(\mathbf{0}) = \int_{\mathbb{R}^n} \Phi(\mathbf{y}) \Delta f(-\mathbf{y}) \, d^n \mathbf{y} = \int_{\mathbb{R}^n} \Phi(\mathbf{y}) \Delta f(\mathbf{y}) \, d^n \mathbf{y}$$

(using the symmetry of Φ to get rid of the minus sign for convenience). We want to use Green's second identity to move the Laplacian to Φ instead. But then we need to restrict attention to a bounded region, and we need to avoid the singularity of Φ at the origin. (It is this singularity that allows the answer to be non-zero, after all.) Pick R sufficiently large so $f(\mathbf{y}) = 0$ for $|\mathbf{y}| \geq R$, so that integrating over $B_R(\mathbf{0})$ instead of \mathbb{R}^n in the integral above does not change its value. Then, thanks to the integrability of Φ near the origin, we find

$$\Delta u(\mathbf{0}) = \lim_{r \rightarrow 0} \int_{B_R(\mathbf{0}) \setminus B_r(\mathbf{0})} \Phi(\mathbf{y}) \Delta f(\mathbf{y}) \, d^n \mathbf{y}.$$

This integral can be transformed by Green's second identity, resulting in

$$\Delta u(\mathbf{0}) = \lim_{r \rightarrow 0} \left(\int_{B_R(\mathbf{0}) \setminus B_r(\mathbf{0})} \Delta \Phi(\mathbf{y}) f(\mathbf{y}) \, d^n \mathbf{y} + \int_{\partial(B_R(\mathbf{0}) \setminus B_r(\mathbf{0}))} (\Phi \partial_n f - f \partial_n \Phi) \, dS \right).$$

The first integral vanishes because $\Delta \Phi = 0$, and in the second integral we can ignore the outer boundary $\partial B_R(\mathbf{0})$ because $f = \partial_n f = 0$ there. Finally, the normal vector n points *inward* on the inner boundary $\partial B_r(\mathbf{0})$, so we get a sign change when we consider the outward point normal instead. Thus we have

$$\Delta u(\mathbf{0}) = \lim_{r \rightarrow 0} \int_{\partial B_r(\mathbf{0})} (f \partial_n \Phi - \Phi \partial_n f) \, dS.$$

The two preceding lemmas then yield $\Delta u(\mathbf{0}) = -f(\mathbf{0})$. ■

Bounded domains and Green's function. Let Ω be a bounded domain with piecewise C^1 boundary. The weak maximum principle (even without the "nice" boundary) immediately shows that the Dirichlet problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega \end{aligned} \tag{DP}$$

has at most one solution for any given data f and g .

Our goal in this section is to generalize the representation formula (9) to this setting. We first concentrate on homogeneous boundary data, i.e., $g = 0$. It will turn out that we get the case for $g \neq 0$ "for free".

First, rewrite (9) to read

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, d^n \mathbf{y}.$$

A similar formula for solutions on Ω might look like

$$u(\mathbf{x}) = \int_{\Omega} G_{\mathbf{y}}(\mathbf{x}) f(\mathbf{y}) \, d^n \mathbf{y}$$

instead. To ensure that this satisfies $u = 0$ on $\partial\Omega$, we want $G_{\mathbf{y}}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. Apart from that, we want $G_{\mathbf{y}}(\mathbf{x})$ (as a function of \mathbf{x}) to be as much "like" $\Phi(\mathbf{x} - \mathbf{y})$ as possible. To make that precise:

Definition. Assume that, for each $\mathbf{y} \in \Omega$, there exists a function $H_{\mathbf{y}} \in C^2(\overline{\Omega})$ which is harmonic in Ω and satisfies

$$H_{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{y}) \quad \text{for } \mathbf{x} \in \partial\Omega.$$

By the maximum principle, this function is unique. Then the function $G_{\mathbf{y}}$ defined by

$$G_{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{y}) - H_{\mathbf{y}}(\mathbf{x})$$

is called the *Green's function* associated with Ω . By construction, this function is C^2 on $\overline{\Omega} \setminus \{\mathbf{y}\}$ and harmonic on $\Omega \setminus \{\mathbf{y}\}$, and it vanishes on $\partial\Omega$.

To discover how this helps us write a solution formula for the Dirichlet problem, we concentrate first on the Φ part.

For simplicity, assume that $\mathbf{y} = \mathbf{0}$. If $\varepsilon > 0$ is small enough, $\overline{B}(\mathbf{0}, \varepsilon) \subset \Omega$. Put $\Omega_{\varepsilon} = \Omega \setminus \overline{B}(\mathbf{0}, \varepsilon)$, and apply Green's second identity to Φ and an arbitrary function $u \in C^2(\overline{\Omega})$:

$$\begin{aligned} \int_{\Omega_{\varepsilon}} (u \Delta \Phi - \Phi \Delta u) \, d^n \mathbf{x} &= \int_{\partial\Omega_{\varepsilon}} (u \partial_n \Phi - \Phi \partial_n u) \, dS \\ &= \int_{\partial\Omega} (u \partial_n \Phi - \Phi \partial_n u) \, dS - \int_{\partial B_{\varepsilon}(\mathbf{0})} (u \partial_n \Phi - \Phi \partial_n u) \, dS, \end{aligned}$$

where the minus sign in front of the last integral is due to the direction of the normal vector \mathbf{n} on $\partial B_\varepsilon(\mathbf{0})$ pointing out of the ball, whereas the normal vector on that part of $\partial\Omega_\varepsilon$ points into the ball.

Now let $\varepsilon \rightarrow 0$. The red term on the left is already zero, while the boundary integral of the red term on the right will vanish in the limit by Lemma 14. Finally, the integral of the green term will converge to $-u(\mathbf{0})$ by Lemma 13, so we end up with the representation formula

$$u(\mathbf{y}) = - \int_{\Omega} \Phi \Delta u \, d^n \mathbf{x} - \int_{\partial\Omega} (u \partial_n \Phi - \Phi \partial_n u) \, dS, \quad \mathbf{y} \in \Omega. \quad (10)$$

(We proved it only for $\mathbf{y} = \mathbf{0}$, but translation invariance ensures that the result generalizes. Or you could rerun the proof with $\Phi(\mathbf{x})$ replaced by $\Phi_{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{y})$ and $B_\varepsilon(\mathbf{0})$ by $B_\varepsilon(\mathbf{y})$.)

Repeat the same calculation with Φ replaced by $H_{\mathbf{y}}$. This is much easier, since $H_{\mathbf{y}}$ is harmonic we do not need to remove a ball: Green's second identity immediately yields

$$\int_{\Omega} (u \Delta H_{\mathbf{y}} - H_{\mathbf{y}} \Delta u) \, d^n \mathbf{x} = \int_{\partial\Omega} (u \partial_n H_{\mathbf{y}} - H_{\mathbf{y}} \partial_n u) \, dS,$$

where again the red term vanishes. Rearranging this into

$$\int_{\Omega} H_{\mathbf{y}} \Delta u \, d^n \mathbf{x} + \int_{\partial\Omega} (u \partial_n H_{\mathbf{y}} - H_{\mathbf{y}} \partial_n u) \, dS = 0$$

and adding it to the right hand side of (10), we obtain

$$u(\mathbf{y}) = - \int_{\Omega} G_{\mathbf{y}} \Delta u \, d^n \mathbf{x} - \int_{\partial\Omega} (u \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n u) \, dS$$

where the red term is zero by construction, so we finally have

$$u(\mathbf{y}) = - \int_{\Omega} G_{\mathbf{y}} \Delta u \, d^n \mathbf{x} - \int_{\partial\Omega} u \partial_n G_{\mathbf{y}} \, dS, \quad \mathbf{y} \in \Omega.$$

We have proved

Theorem 16. *If $\Omega \subset \mathbb{R}^n$ is a bounded region with C^1 boundary, and if Ω admits a Green's function G , then the solution $u \in C^2(\overline{\Omega})$ of (DP), if it exists, is given by*

$$u(\mathbf{y}) = \int_{\Omega} G_{\mathbf{y}}(\mathbf{x}) f(\mathbf{x}) \, d^n \mathbf{x} - \int_{\partial\Omega} \partial_n G_{\mathbf{y}}(\mathbf{x}) g(\mathbf{x}) \, dS(\mathbf{x}), \quad \mathbf{y} \in \Omega. \quad (11)$$

Notice the careful wording: We have not shown that the Dirichlet problem has a solution. We have only shown what it must be, if it solves the problem.

We now show that Green's function is symmetric.

Proposition 17. *Assume that Ω is a bounded region with piecewise C^1 boundary, and which admits a Green's function. Then the Green's function G for this region satisfies $G_{\mathbf{y}}(\mathbf{x}) = G_{\mathbf{x}}(\mathbf{y})$, whenever $\mathbf{x}, \mathbf{y} \in \Omega$.*

Proof. First a "physicists' proof" to reveal the essentials: Since $G_{\mathbf{x}}$ vanishes on $\partial\Omega$ and $-\Delta G_{\mathbf{x}}(z) = \delta(z - \mathbf{x})$, and similarly for $G_{\mathbf{y}}$, Green's second identity yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} (G_{\mathbf{x}} \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n G_{\mathbf{x}}) \, dS = \int_{\Omega} (G_{\mathbf{x}} \Delta G_{\mathbf{y}} - G_{\mathbf{y}} \Delta G_{\mathbf{x}}) \, d^n z \\ &= \int_{\Omega} (-G_{\mathbf{x}}(z) \delta(z - \mathbf{y}) + G_{\mathbf{y}}(z) \delta(z - \mathbf{x})) \, d^n z = -G_{\mathbf{x}}(\mathbf{y}) + G_{\mathbf{y}}(\mathbf{x}). \end{aligned}$$

Our problem with this proof is that Green's second identity, as we know it, requires C^2 functions, which is not the case here. So we remove some small balls around the troublesome points \mathbf{x} and \mathbf{y} : Put $\Omega_\varepsilon = \Omega \setminus (\overline{B_\varepsilon(\mathbf{x})} \cup \overline{B_\varepsilon(\mathbf{y})})$, where $\varepsilon > 0$ is small enough so that the two removed balls lie within Ω and have no point in common.

Now $\Delta G_{\mathbf{x}}$ and $\Delta G_{\mathbf{y}}$ are harmonic in Ω_ε , so Green's theorem is applicable:

$$\begin{aligned} 0 &= \int_{\partial\Omega_\varepsilon} (G_{\mathbf{x}} \Delta G_{\mathbf{y}} - G_{\mathbf{y}} \Delta G_{\mathbf{x}}) \, d^n z = \int_{\partial\Omega_\varepsilon} (G_{\mathbf{x}} \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n G_{\mathbf{x}}) \, dS \\ &= - \left(\int_{\partial B_\varepsilon(\mathbf{x})} + \int_{\partial B_\varepsilon(\mathbf{y})} \right) (G_{\mathbf{x}} \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n G_{\mathbf{x}}) \, dS, \end{aligned}$$

where we dropped the integral over $\partial\Omega$, because the integrand is zero there. The minus sign is due to the reversed direction of the normal vector on the boundary of the balls versus the boundary of Ω_ε . We have already dealt with this sort of boundary integral in the proof of Theorem 16, so here we go again! Look at the first of the two boundary integrals above:

$$\int_{\partial B_\varepsilon(\mathbf{x})} (G_{\mathbf{x}} \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n G_{\mathbf{x}}) \, dS.$$

The integral of the red term is zero, for $G_{\mathbf{x}}$ is constant on $\partial B_\varepsilon(\mathbf{x})$, and

$$\int_{\partial B_\varepsilon(\mathbf{x})} \partial_n G_{\mathbf{y}} \, dS = \int_{B_\varepsilon(\mathbf{x})} \Delta G_{\mathbf{y}} \, d^n z = 0$$

by the divergence theorem. Even without harmonicity, this would vanish in the limit as $\varepsilon \rightarrow 0$.

For the green term, note that $-\partial_{\mathbf{y}} G_{\mathbf{x}} = 1/(A_n \varepsilon^{n-1})$ on $\partial B_\varepsilon(\mathbf{x})$, so together with the above calculation we get

$$\int_{\partial B_\varepsilon(\mathbf{x})} (G_{\mathbf{x}} \partial_n G_{\mathbf{y}} - G_{\mathbf{y}} \partial_n G_{\mathbf{x}}) \, dS = \int_{\partial B_\varepsilon(\mathbf{x})} G_{\mathbf{y}} \, dS \rightarrow G_{\mathbf{y}}(\mathbf{x}) \quad \text{as } \varepsilon \rightarrow 0.$$

Similarly,

$$\int_{\partial B_\varepsilon(y)} (G_x \partial_n G_y - G_y \partial_n G_x) dS = \int_{\partial B_\varepsilon(y)} G_x dS \rightarrow G_x(y) \quad \text{as } \varepsilon \rightarrow 0,$$

and the proof is complete. ■

Remark. A different take on the proof of symmetry of Green's function:

Let $f, g \in C(\bar{\Omega})$, and let u, v solve $-\Delta u = f$ and $-\Delta v = g$ in Ω , $u = v = 0$ on $\partial\Omega$. Thus

$$u(y) = \int_{\Omega} G_y(x) f(x) d^n x, \quad v(y) = \int_{\Omega} G_y(x) g(x) d^n x.$$

Since u and v vanish on $\partial\Omega$, Green's second identity applied to u and v implies

$$\int_{\Omega} u(y)g(y) d^n y = \int_{\Omega} v(y)f(y) d^n y.$$

Plugging in the above formulas for u and v , swapping the integration variables x and y in one of the integrals, and rearranging, we get

$$\int_{\Omega} \int_{\Omega} (G_y(x) - G_x(y))f(x)g(y) d^n x d^n y = 0,$$

from which the desired conclusion quickly follows. (Take f and g to be approximate delta functions concentrated near two different points in Ω .)

It may be interesting to study the above argument from an elementary linear algebra point of view. The last part resembles the proof that, if A is a quadratic matrix satisfying $y^T A x = x^T A y$ for all vectors x and y , then A is symmetric. (Just take $x = e_i$ and $y = e_j$.) The first part resembles the proof that the inverse of an invertible symmetric matrix is symmetric. Think of the "matrix" $G_x(y)$ (where x and y play the rôles of matrix indices) as the inverse of the operator $-\Delta$. And the symmetry of this operator is just the identity $\int_{\Omega} u \Delta v = \int_{\Omega} v \Delta u$. (Green's second identity, when both functions vanish on the boundary.)

Green's function for balls. Here we compute Green's function for the open unit ball $B^n = B_1(0)$ in \mathbb{R}^n . I will drop some exponents and write B instead of B^n and S instead of S^{n-1} for the unit sphere ∂B .

If $y \in B$, we need to find a harmonic function on B with the same values as $\Phi(x - y)$ for $x \in S$. The trick is to "put a charge" at a suitable point w outside B . It turns out that $w = y/|y|^2$ does the trick. Since $|x| = 1$, we find

$$|x - y|^2 = 1 + |y|^2 - 2x \cdot y$$

and

$$|x - w|^2 = 1 + |w|^2 - 2x \cdot w = 1 + \frac{1}{|y|^2} - 2 \frac{x \cdot y}{|y|^2} = \frac{|x - y|^2}{|y|^2},$$

so that

$$|y| |x - w| = |x - y| \quad \text{when } |x| = 1.$$

Therefore, we can pick $H_y(x) = \Phi(|y|(x - w))$,⁴ resulting in

$$G_y(x) = \Phi(x - y) - \Phi(|y|(x - w)).$$

Here we may note that

$$|y|^2 |w - x|^2 = |y|^2 (|w|^2 + |x|^2 - 2x \cdot w) = 1 + |x|^2 |y|^2 - 2x \cdot y,$$

so we may also write this as

$$G_y(x) = \Phi(x - y) - \Phi\left(\sqrt{1 + |x|^2 |y|^2 - 2x \cdot y}\right)$$

which is indeed symmetric in x, y as expected.

To use the solution formula (11), we need to compute $-\partial_n G_y(x)$ for $y \in B$ and $x \in S$, i.e., for $|y| < 1$ and $|x| = 1$. We find

$$-\nabla G_y(x) = \frac{x - y}{A_n |x - y|^n} - \frac{|y|^2 (x - w)}{A_n (|y| |x - w|)^n}.$$

When $|x| = 1$, the two denominators coincide, so we end up with

$$P_y(x) := -x \cdot \nabla G_y(x) = \frac{1 - |y|^2}{A_n |x - y|^n}.$$

where we have named the result $P_y(x)$, the (n -dimensional) *Poisson kernel*

Summarising what we have found, any continuous function on \bar{B} satisfying $-\Delta u = f$ in B and $u = g$ on ∂B is given by (see (11))

$$u(y) = \int_B G_y(x) f(x) d^n x + \int_S P_y(x) g(x) dS(x), \quad y \in \Omega.$$

The special case where $f = 0$ is known as *Poisson's integral formula* for harmonic functions on the ball.

⁴An earlier version of this note did not use the obvious trick of scaling the *argument* of Φ , thus leading to a more involved computation.

Now consider the special case $n = 2$. If we put $r = |y|$ and let θ be the angle between the two vectors x and y , we get the 2-dimensional *Poisson kernel*

$$P_r(\theta) := P_y(x) = \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta}$$

(in the hope that the two uses of the letter P will not lead to confusion).

Referring back to (11), the solution to the Dirichlet problem $-\Delta u = 0$ on \mathbb{D} with $u(\cos\theta, \sin\theta) = g(\theta)$ should then be given by *Poisson's integral formula*

$$u(r\cos\varphi, r\sin\varphi) = \int_{-\pi}^{\pi} P_r(\varphi-\theta)g(\theta) d\theta.$$

It is essentially the same as the higher-dimensional case, but looks different because of the special structure of the unit circle: We have taken advantage of the ability to add and subtract angles. In fancier language, the unit circle is a group, whereas higher dimensional spheres are mostly⁵ not.

It was originally derived using separation of variables and a Fourier analysis. For this, we may note that the Laplace operator in polar coordinates takes the form

$$\Delta u = r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta}.$$

We look for harmonic functions of the form $u = R(r)\Theta(\theta)$, resulting in $r^{-1}(rR')'\Theta + r^{-2}R\Theta'' = 0$. After separating the variables, we are left with $\Theta'' = -\lambda\Theta$ and $(rR')' = \lambda r^{-1}R$. Since Θ must be 2π -periodic, we must put $\lambda = n^2$ for an integer n , so the R equation is $r(rR')' = n^2R$. We try $R = r^k$, and get $rR' = kr^k$, $r(rR')' = k^2r^k$, with the non-trivial solutions $k = \pm n$. Rejecting solutions with negative k (which have a singularity at $r = 0$), we are therefore left with $R = r^{|n|}$. Thus we are led to look for solutions of the form

$$u(r\cos\varphi, r\sin\varphi) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\varphi}.$$

Matching this to $g(\theta)$ at $r = 1$, we should have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta,$$

and therefore

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} g(\theta) r^{|n|} e^{in(\varphi-\theta)} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) P(r, \varphi-\theta) d\theta,$$

⁵The exception being S^3 , the unit quaternions.

where

$$P(r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Here we note that

$$\sum_{k=0}^{\infty} r^k e^{\pm ik\theta} = \frac{1}{1 - re^{\pm ik\theta}},$$

so

$$\begin{aligned} P(r, \theta) &= \frac{1}{2\pi} \left(\frac{1}{1 - re^{ik\theta}} + \frac{1}{1 - re^{-ik\theta}} - 1 \right) \\ &= \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta}, \end{aligned}$$

which we recognize as the Poisson kernel introduced above.

Theorem 18 (Harnack's inequality). *If $u \geq 0$ is continuous on $\overline{B_R(0)}$ and harmonic in $B_R(0) \subset \mathbb{R}^n$, then*

$$\frac{R-|x|}{(R+|x|)^{n-1}} R^{n-2} u(0) \leq u(x) \leq \frac{R+|x|}{(R-|x|)^{n-1}} R^{n-2} u(0)$$

when $|x| < R$.

Proof. If $R = 1$, use the inequalities $1 - |x| < |x - y| < 1 + |x|$ (when $|y| = 1$) in Poisson's formula. The case $R \neq 1$ can be reduced to the case $R = 1$ by rescaling. The details are left to the reader. ■

The Dirichlet problem on a ball. The boundary value problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

is known as the *Dirichlet problem* for the region Ω .

Simple properties of the Poisson kernel show that the Dirichlet problem for B^n ($n \geq 2$) has a solution for all $g \in C(\partial B)$.

These properties, easily verified, are:

- $P_y(x)$ is a harmonic function of y for any $x \in S$,
- $P_y(x) > 0$ for all $y \in B$ and $x \in S$,

- $\int_S P_y(\mathbf{x}) \, dS(\mathbf{x}) = 1$, and
- $\lim_{B \ni \mathbf{y} \rightarrow \mathbf{z}} \int_S P_y(\mathbf{x}) [|\mathbf{x} - \mathbf{z}| > \delta] \, dS(\mathbf{x}) = 0$ for all $\mathbf{x}, \mathbf{z} \in S$ and $\delta > 0$.

The claim is now that for any $g \in C(S)$, the function

$$u(\mathbf{y}) = \int_S P_y(\mathbf{x}) g(\mathbf{x}) \, dS(\mathbf{x})$$

solves the Dirichlet problem. It is clearly harmonic. Further, for any $\mathbf{z} \in S$, we find

$$u(\mathbf{y}) - g(\mathbf{z}) = \int_S P_y(\mathbf{x}) (g(\mathbf{x}) - g(\mathbf{z})) \, dS(\mathbf{x}),$$

and so

$$|u(\mathbf{y}) - g(\mathbf{z})| \leq \int_S P_y(\mathbf{x}) |g(\mathbf{x}) - g(\mathbf{z})| \, dS(\mathbf{x}).$$

Given $\varepsilon > 0$, pick $\delta > 0$ so that $|\mathbf{x} - \mathbf{z}| < \delta$ implies $|g(\mathbf{x}) - g(\mathbf{z})| < \varepsilon$. Divide the integral above in two pieces, one where $|\mathbf{x} - \mathbf{z}| < \delta$ and the rest, and conclude that the integral can be made smaller than 2ε by letting \mathbf{y} be sufficiently close to \mathbf{z} .

The Dirichlet problem on bounded domains. Before we get to the meat of the matter, we need to develop some tools. First, we generalise our definitions of sub- and superharmonic functions:

Definition. A continuous function f on a domain Ω is called *subharmonic* if for each $\mathbf{x} \in \Omega$

$$f(\mathbf{x}) \leq \int_{\partial B_r(\mathbf{x})} f(\mathbf{y}) \, dS(\mathbf{y}) \tag{12}$$

holds for all sufficiently small $r > 0$. It is called *superharmonic* if the opposite inequality holds.

These notions extend to lower semicontinuous functions for the subharmonic case, and upper semicontinuous functions for the superharmonic case. However, we shall have no need for this refinement.

Lemma 19. *The weak and strong maximum principles hold for any subharmonic function. When f is subharmonic, the inequality (12) holds whenever $B_r(\mathbf{x}) \subset \Omega$.*

Proof. Inspect the proof of Theorem 7. The only property required of the proof is (12), written in that proof as $\tilde{f}_x(r) \geq f(\mathbf{x})$, for all \mathbf{x} and sufficiently small r (but how small is allowed to depend on \mathbf{x}). Therefore, the maximum principle holds.

To show that the defining inequality holds for all r , so long as $\bar{B}_r(\mathbf{x}) \subset \Omega$, let $u \in C(\bar{B}_r(\mathbf{x}))$ be harmonic in $B_r(\mathbf{x})$ and equal to f on $\partial B_r(\mathbf{x})$. The maximum principle applied to the subharmonic function $f - u$ implies $f - u \leq 0$, so $f(\mathbf{x}) \leq u(\mathbf{x}) = \int_{B_r(\mathbf{x})} f \, dS$. ■

We shall write $f_1 \vee f_2$ and $f_1 \wedge f_2$ for the *pointwise maximum* and *pointwise minimum*, respectively, of two functions f_1, f_2 :

$$(f_1 \vee f_2)(\mathbf{x}) = \max(f_1(\mathbf{x}), f_2(\mathbf{x})), \quad (f_1 \wedge f_2)(\mathbf{x}) = \min(f_1(\mathbf{x}), f_2(\mathbf{x})).$$

Proposition 20. *If f_1 and f_2 are subharmonic, then so is $f_1 \vee f_2$. If f_1 and f_2 are superharmonic, then so is $f_1 \wedge f_2$.*

Proof. To prove the first statement, note that

$$f_1(\mathbf{x}) \leq \int_{\partial B_r(\mathbf{x})} f_1(\mathbf{y}) \, dS(\mathbf{y}) \leq \int_{\partial B_r(\mathbf{x})} (f_1 \vee f_2)(\mathbf{y}) \, dS(\mathbf{y}),$$

since f_1 is subharmonic and $f_1 \leq f_1 \vee f_2$. The same result holds with f_1 and f_2 interchanged, and the result follows.

The second statement follows from the first by multiplying by -1 . ■

Definition. If $f \in C(\Omega)$ is subharmonic and B is an open ball with $\bar{B} \subset \Omega$, the *harmonic lifting* of f on B is the function $\tilde{f} \in C[\Omega]$ which equals f outside B and is harmonic inside B . (In other words, it solves the Dirichlet problem in B with boundary values $f|_{\partial\Omega}$.)

Proposition 21. *The harmonic lifting \tilde{f} of a subharmonic function f is itself subharmonic, and $\tilde{f} \geq f$.*

Proof. The inequality follows inside B by applying the maximum principle to $f - \tilde{f}$ (which is subharmonic in B , and zero on ∂B). Outside B , it is an equality.

To show that \tilde{f} is subharmonic, consider any $\mathbf{x} \in \Omega$. If $\mathbf{x} \notin B$, then

$$\tilde{f}(\mathbf{x}) = f(\mathbf{x}) \leq \int_{\partial B_r(\mathbf{x})} f(\mathbf{y}) \, dS(\mathbf{y}) \leq \int_{\partial B_r(\mathbf{x})} \tilde{f}(\mathbf{y}) \, dS(\mathbf{y}).$$

On the other hand, if $\mathbf{x} \in B$, then $B_r(\mathbf{x}) \subset B$ for any sufficiently small $r > 0$. Since \tilde{f} is harmonic in B ,

$$\tilde{f}(\mathbf{x}) = \int_{\partial B_r(\mathbf{x})} \tilde{f}(\mathbf{y}) \, dS(\mathbf{y}),$$

and the proof is complete. ■

It is now time to present our candidate for a solution to the Dirichlet problem on a general bounded region.

Theorem 22. *Let $\Omega \subset \mathbb{R}^n$ be a bounded region, and let $g \in C(\partial\Omega)$. Call a function $f \in C(\overline{\Omega})$ a subsolution if f is subharmonic in Ω and $f \leq g$ on $\partial\Omega$, and define $u : \Omega \rightarrow \mathbb{R}$ by*

$$u(\mathbf{x}) = \sup \{ f(\mathbf{x}) \mid f \text{ is a subsolution} \}.$$

Then u is harmonic in Ω .

Proof. First of all, note that since g is bounded, subsolutions do exist (take any constant $\leq \min_{\partial\Omega} g$). Also, any subsolution is bounded above by $\max_{\partial\Omega} g$, so u is a well defined and bounded function.

Fix some point $\mathbf{z} \in \Omega$, and numbers $0 < r < R$ with $\overline{B}_R(\mathbf{z}) \subset \Omega$. We shall show that u is harmonic on $B_r(\mathbf{z})$. Clearly, this is sufficient.

Now take a sequence of subsolutions f_k so that $\lim_{k \rightarrow \infty} f_k(\mathbf{z}) = u(\mathbf{z})$, and for each k let v_k be the harmonic lifting of $f_1 \vee f_2 \vee \dots \vee f_k$ to $B_R(\mathbf{z})$. Then v_k is a subsolution (use Propositions 20 and 21), and since $v_k \geq f_1 \vee f_2 \vee \dots \vee f_k$, we also find that $\lim_{k \rightarrow \infty} v_k(\mathbf{z}) = u(\mathbf{z})$.

Moreover, since $v_k \leq v_{k+1}$ for all k , the pointwise limit $v := \lim_{k \rightarrow \infty} v_k$ exists. Also $v \leq u$ since each $v_k \leq u$ (being a subsolution). We now make two claims. Taken together, they finish the proof.

Claim 1: v is harmonic on $B_r(\mathbf{z})$.

Claim 2: $v = u$ on $B_r(\mathbf{z})$

We prove both claims using Harnack's inequality (Theorem 18). In the current context, it implies the following: *For any function $f \in C(\overline{B}_R(\mathbf{z}))$ which is harmonic and non-negative in $B_R(\mathbf{z})$,*

$$f(\mathbf{x}) \leq \frac{(R+r)R^{n-2}}{(R-r)^{n-1}} f(\mathbf{z}) \quad (\mathbf{x} \in B_r(\mathbf{z})). \quad (13)$$

To prove *Claim 1*, we show that $v_k \rightarrow v$ uniformly on $B_r(\mathbf{z})$. This is trivially accomplished by applying (13) to $v_l - v_k$ where $k < l$, and then letting $l \rightarrow \infty$, resulting in

$$v(\mathbf{x}) - v_k(\mathbf{x}) \leq \frac{(R+r)R^{n-2}}{(R-r)^{n-1}} (v(\mathbf{z}) - v_k(\mathbf{z})) \quad (\mathbf{x} \in B_r(\mathbf{z})),$$

noting that the right hand side converges to zero and does not depend on \mathbf{x} .

To prove *Claim 2*, assume the opposite. Then there is some $\mathbf{x} \in B_r(\mathbf{z})$ with $v(\mathbf{x}) < u(\mathbf{x})$. So there is some subsolution f_0 with $f_0(\mathbf{x}) > v(\mathbf{x})$. Now define v_k to be the

harmonic lifting of $f_0 \vee f_1 \vee \dots \vee f_k$ to $B_R(\mathbf{z})$. Go through the first part of the proof, with v_k replaced by w_k . The limit w is harmonic, $v \leq w \leq u$, and $w(\mathbf{x}) \geq f_0(\mathbf{x}) > v(\mathbf{x})$. However, since $v(\mathbf{z}) = u(\mathbf{z})$, we must have $w(\mathbf{z}) = v(\mathbf{z})$. Now apply (13) to $w - v$. This immediately leads to a contradiction, since the right hand side is zero while the left hand side is positive. Thus *Claim 2* must hold. ■

Unfortunately, the function u produced by this theorem does not necessarily obey the given boundary conditions. For example, let $\Omega = \mathbb{D} \setminus \{0\} \subset \mathbb{R}^2$ (here $\mathbb{D} = B_1(\mathbf{0})$), and let the boundary value g be given as 0 on $\partial\mathbb{D}$ and 1 on the remaining boundary point $\mathbf{0}$. Clearly, the constant 0 is a subsolution, and (not quite as clear, but true) $1 \wedge (-\varepsilon \ln|x|)$ is a supersolution (see below) for $\varepsilon > 0$. Since no subsolution can be greater, for any $\varepsilon > 0$, every subsolution is ≤ 0 . It follows that $u = 0$. It satisfies the boundary condition only on $\partial\mathbb{D}$, but not at the isolated boundary point.

A *supersolution* is a function continuous on $\overline{\Omega}$ and superharmonic in Ω which is $\geq g$ on $\partial\Omega$. If f_1 is a subsolution and f_2 is a supersolution, the maximum principle applied to the subharmonic function $f_1 - f_2$ shows that $f_1 \leq f_2$.

We need a sufficient supply of super- and subsolutions to force u to have the right boundary values.

Definition. A *barrier* at $\mathbf{y} \in \partial\Omega$ is a function $h \in C(\overline{\Omega})$ which is superharmonic in Ω with $h(\mathbf{y}) = 0$ and $h > 0$ on $\partial\Omega \setminus \{\mathbf{y}\}$. The boundary point \mathbf{y} is called *regular* if there exists a barrier at \mathbf{y} . Further, we say that $\partial\Omega$ (or a subset) is *regular* if all its points are regular boundary points.

Further, a *local barrier* at \mathbf{y} is a function $h \in C(\overline{\Omega} \cap \overline{B})$ which is superharmonic in $\Omega \cap B$, with $h(\mathbf{y}) = 0$ and $h > 0$ on $(\partial B \cap \overline{\Omega}) \cup (\overline{B} \cap \partial\Omega) \setminus \{\mathbf{y}\}$. Here B is a ball centered at \mathbf{y} .

It turns out that the existence of a local barrier is sufficient to prove regularity of a boundary point. A rough proof outline is as follows: Let $0 < \varepsilon < \min \{ h(\mathbf{x}) \mid \mathbf{x} \in \partial B \cap \overline{\Omega} \}$. Let $H = h \wedge \varepsilon$ in $\overline{\Omega} \cap \overline{B}$, and $H = \varepsilon$ on the rest of $\overline{\Omega}$. Then H is a barrier at \mathbf{y} .

Lemma 23. *In the setting of Theorem 22, assume that \mathbf{y} is a regular boundary point of Ω . Then $\lim_{\mathbf{x} \rightarrow \mathbf{y}} u(\mathbf{x}) = g(\mathbf{y})$.*

Proof. Let $\varepsilon > 0$. Pick $\delta > 0$ so that $\mathbf{x} \in \partial\Omega$ and $|\mathbf{x} - \mathbf{y}| < \delta$ imply $|g(\mathbf{x}) - g(\mathbf{y})| < \varepsilon$. Let $m > 0$ be the minimum value of h on the set $\{\mathbf{x} \in \partial\Omega \mid |\mathbf{x} - \mathbf{y}| \geq \delta\}$, and let M be the maximum value of $|g(\mathbf{x}) - g(\mathbf{y})|$ on the same set. Define functions f_+ and f_- by

$$f_{\pm}(\mathbf{x}) = g(\mathbf{y}) \pm \varepsilon \pm \frac{M}{m} h(\mathbf{x}).$$

Then f_+ is a supersolution and f_- is a subsolution, and so $f_- \leq u \leq f_+$. Since $\lim_{\mathbf{x} \rightarrow \mathbf{y}} f_{\pm}(\mathbf{x}) = g(\mathbf{y}) \pm \varepsilon$, there is some $\delta' > 0$ so that $\mathbf{x} \in \Omega$ and $|\mathbf{x} - \mathbf{y}| < \delta'$ imply $|u(\mathbf{x}) - g(\mathbf{y})| < 2\varepsilon$. ■

Theorem 24. *The Dirichlet problem on a bounded region $\Omega \subset \mathbb{R}^n$ has a (unique) solution for every continuous boundary value if and only if its boundary is regular.*

Proof. The uniqueness of any solution is a consequence of the maximum principle. The existence of a solution, if the boundary is regular, follows from Lemma 23. On the other hand, assume every Dirichlet problem with a continuous boundary value has a solution. For any $y \in \partial\Omega$, let $g(x) = |x - y|$ for $x \in \partial\Omega$, and let u be the solution to the corresponding Dirichlet problem. Then u is a barrier at y , hence y is regular. ■

Proposition 25. *For a given $y \in \partial\Omega$, assume the existence of a ball $B \subset \mathbb{R}^n$ with $\bar{B} \cap \bar{\Omega} = \{y\}$. (This is called the exterior ball condition.) Then y is a regular boundary point.*

Proof. The function h defined by

$$h(x) = \Phi(y - z) - \Phi(x - z),$$

where z is the center of B , is a barrier. ■

There is a refinement to the exterior ball condition, *the exterior cone condition*, where the ball is replaced by a truncated cone with just the tip of the cone touching $\partial\Omega$. We shall not prove it here.

However, in \mathbb{R}^2 we can do much better: We say a boundary point y of $\Omega \subset \mathbb{R}^2$ can be reached by a path from the exterior if there exists a simple continuous path γ with one end point at y , and the rest of the path lying outside $\bar{\Omega}$.

Proposition 26. *A boundary point of a region $\Omega \subset \mathbb{R}^2$ is regular if it can be reached by a path from the exterior.*

Proof. We lose no generality by assuming that the given boundary point is placed at the origin. Think of \mathbb{R}^2 as the complex plane \mathbb{C} , write $z = x + iy = re^{i\theta}$, and put

$$h(x, y) = -\operatorname{Re} \frac{1}{\ln z} = -\operatorname{Re} \frac{1}{\ln r + i\theta} = -\operatorname{Re} \frac{\ln r - i\theta}{(\ln r)^2 + \theta^2} = \frac{-\ln r}{(\ln r)^2 + \theta^2}.$$

Then h , being the real part of a complex analytic function, is harmonic. It is positive for $r < 1$ and converges to zero as $(x, y) \rightarrow (0, 0)$, so it is a local barrier.

There is just *one small problem* with the above, namely, that the argument θ may not be well defined, since $\theta + 2k\pi$ represents the same angle as θ for any integer k . This is where the path γ comes in: It allows us to have θ being a well defined continuous function on the complement of γ , at least on some neighbourhood of the origin. (The neighbourhood has to be small enough for γ to leave it.) If you were to cross the path γ , θ would jump by 2π . But since this is outside $\bar{\Omega}$, that is not a problem. ■

We conclude with an example, due to Lebesgue, of a domain in \mathbb{R}^3 with a non-regular boundary point. To this end, define

$$u(x, y, z) = \int_0^1 \frac{s \, ds}{\sqrt{x^2 + y^2 + (z + s)^2}} \quad \text{in } \mathbb{R}^2 \setminus \{0\} \times \{0\} \times [-1, 0].$$

This is harmonic, since the integrand is a harmonic function of (x, y, z) . It may help to think of it as the resulting potential due to mass of linear density $|z|$ (the factor s in the numerator) distributed along the line segment $\{0\} \times \{0\} \times [-1, 0]$. It is the fact that this density goes to zero at the origin which makes the example work. Writing $r = (x^2 + y^2)^{1/2}$, we compute

$$\begin{aligned} u &= \int_0^1 \frac{(z + s) \, ds}{\sqrt{r^2 + (z + s)^2}} - \int_0^1 \frac{z \, ds}{\sqrt{r^2 + (z + s)^2}} \\ &= \left[\sqrt{r^2 + (z + s)^2} - z \operatorname{asinh} \frac{z + s}{r} \right]_{s=0}^{s=1} \end{aligned}$$

where the inverse hyperbolic sine function is

$$\operatorname{asinh} t = \ln(t + \sqrt{t^2 + 1}).$$

Substituting that in, simplifying and reordering a bit, we end up with

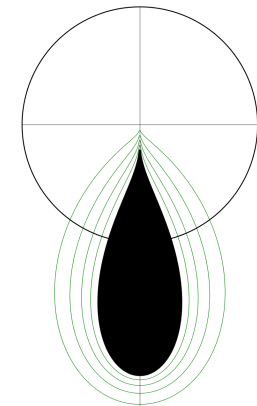
$$u = \sqrt{r^2 + (z + 1)^2} - \sqrt{r^2 + z^2} - z \ln(z + 1 + \sqrt{r^2 + (z + 1)^2}) + z \ln(z + \sqrt{r^2 + z^2}).$$

We are interested in the local behaviour of this near the origin. The first two terms are continuous everywhere, and so is the third (except on $\{0\} \times \{0\} \times (-\infty, -1]$), while the final, red term approaches infinity on the negative z axis $\{0\} \times \{0\} \times (-\infty, 0)$. Since also $u = (r + (r^2 + 1)^{1/2})$ when $z = 0$, it follows that u takes every value greater than 1 in every neighbourhood of the origin (with $z < 0$). In particular, u has no limit at the origin. Define

$$\Omega = \{(x, y, z) \mid u(x, y, z) < 2 \text{ and } x^2 + y^2 + z^2 < \frac{1}{4}\}.$$

See the picture on the right: The z axis is vertical, and the r axis is horizontal. (I cheated a bit, plotting negative r as well, to emphasise the symmetry. The picture should really be rotated about the z axis.) The black region is defined by $u > 2$, and the green lines are level curves for equally spaced values between 1 and 2. Note that the level curves should all be tangent to the z axis at the origin; the plotting software did not quite manage this. The region Ω , then, is the interior of the ball with the black part removed.

By construction, u is continuous on $\partial\Omega$, for it is constantly equal to 2 on the part of $\partial\Omega$ inside the ball. Assume now that $v \in C(\bar{\Omega})$ solves the Dirichlet problem with boundary values given by u . By construction, $0 < u \leq 2$ in $\bar{\Omega}$, so the same must be true of v , by the maximum



principle. Put $w = u - v$; then w is harmonic in Ω , $|w| \leq 2$, and $w \in C(\overline{\Omega} \setminus \{0\})$ with $w = 0$ on $\partial\Omega \setminus \{0\}$. Consider $\Omega_\varepsilon := \Omega \setminus \overline{B}_\varepsilon(\mathbf{0})$. On $\partial\Omega_\varepsilon$, we have the estimate

$$|w(x, y, z)| \leq \frac{2\varepsilon}{\sqrt{r^2 + z^2}},$$

for the right hand side equals 2 on $\partial B_\varepsilon(\mathbf{0})$, and $w(x, y, z) = 0$ on the rest of $\partial\Omega_\varepsilon$. Since w and the right hand side both are harmonic, this inequality holds in Ω_ε as well by the maximum principle. Since this holds for any $\varepsilon > 0$, and since any point in Ω belongs to Ω_ε when ε is small enough, $w = 0$ in Ω , so $u = v$ in Ω . But u has no limit at the origin, whereas v does. This is a contradiction.

Notation used in this document. The partial derivative of a function f with respect to s is more often than not shortened from the conventional $\partial f / \partial s$ to $\partial_s f$. The outer unit normal vector of a region Ω with C^1 boundary is written \mathbf{n} .⁶ The *normal derivative* of a C^1 function u on $\overline{\Omega}$ is written $\partial_n u = \mathbf{n} \cdot \nabla u$. The Laplacian of u is written $\Delta u = \nabla \cdot \nabla u$. We use the “barred” integral sign \int to indicate an *average*, defined to be the ordinary integral divided by the total measure (length, area, or volume) of the region of integration. Thus no matter what we integrate over, $\int_D 1 \, dx = 1$. We write A_n for the area of the unit sphere S^{n-1} of \mathbb{R}^n . Thus a sphere of radius r has area $A_n r^{n-1}$, and the volume of a ball of radius r in \mathbb{R}^n (obtained by integrating $A_n r^{n-1}$) is $A_n r^n / n$. The *Iverson bracket* $[S]$ has the value 1 if the statement S inside is true, and 0 otherwise.

⁶The Greek letter ν is commonly used for this, but in the font used in this note, that is almost indistinguishable from the latin letter ν .