# Harmonicfunctionology

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The Laplace operator on $\mathbb{R}^n$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

plays a rôle in the wave and heat equations, but even more fundamentally, in the *Laplace equation*

$$\Delta u = 0 \tag{1}$$

and the *Poisson equation* $-\Delta u = f$ where $f$ is a given function.

A $C^2$ solution of (1) is called *harmonic*. (Later, in Theorem 4, we will find that harmonic functions are in fact infinitely differentiable.)

Much is gained from the trivial observation that $\Delta u = \nabla\cdot\nabla u$ together with various applications of the divergence theorem or two of its corollaries, Green's first and second identities.

Let's get started by simply integrating $\Delta u$ over a bounded domain $\omega$ with piecewise $C^1$ boundary:[1]

$$\int_\omega \Delta u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_\omega \nabla\cdot\nabla u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\omega} \partial_n u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}). \tag{2}$$

This immediately proves

**Proposition 1.** *If a $C^2$ function $u$ on a domain $\Omega$ is harmonic, then*

$$\int_{\partial\omega} \partial_n u\,\mathrm{d}S = 0 \tag{3}$$

*for all bounded domains $\omega$ with $\overline{\omega}\subset\Omega$ having piecewise $C^1$ boundary.*

*Conversely, if (3) holds for every ball $\omega = B(\boldsymbol{x},r)$ whose closure lies within $\Omega$, then $u$ is harmonic.*

*Proof.* We have already proved the first part. For the converse, (3) and (2) imply that the *average* of $\Delta u$ over any ball is zero. By letting the radius of the ball $B(\boldsymbol{x},r)$ tend to zero, we conclude that $\Delta u(\boldsymbol{x}) = 0$. ∎

**Definition.** The (*radius r*) *spherical average* of a function $u$ at a point $\boldsymbol{x}$ is defined to be

$$\bar{u}_{\boldsymbol{x}}(r) = ⨍_{\partial B(\boldsymbol{x},r)} u\,\mathrm{d}S = ⨍_{\mathbb{S}^{n-1}} u(\boldsymbol{x}+r\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

[1] Notation used here and elsewhere is explained at the end of this note.





where $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ is the unit sphere and the "barred" integral signs denote the average:

$$⨍_{\partial B(\boldsymbol{x},r)} u\,\mathrm{d}S = \frac{1}{A_n r^{n-1}}\int_{\partial B(\boldsymbol{x},r)} u\,\mathrm{d}S,$$

and $A_n$ is the area of $\mathbb{S}^{n-1}$. Note that the second integral in the definition of $\bar{u}_{\boldsymbol{x}}(r)$ makes sense even for $r<0$; thus, we adopt this as the definition for all real $r$ for which the integrand is defined on $\mathbb{S}^{n-1}$. We see that $\bar{u}_{\boldsymbol{x}}$ is an *even* function; it is $C^k$ if $u$ is $C^k$, and $\bar{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$.

When $\omega$ is a ball, we can rewrite (2) in terms of spherical averages: Noting that the volume of the ball $B(\boldsymbol{x},r)$ is $A_n r^n/n$, we find

$$⨍_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = \frac{n}{r}⨍_{B(\boldsymbol{x},r)} \partial_n u(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = \frac{n}{r}⨍_{\mathbb{S}^{n-1}} \partial_r u(\boldsymbol{x}+r\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

where we can move the $r$ derivative outside the integral, and arrive at

$$⨍_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = \frac{n}{r}\bar{u}'_{\boldsymbol{x}}(r). \tag{4}$$

Along with $\bar{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$, this implies

**Theorem 2** (The mean value property of harmonic functions)**.** *A $C^2$ function $u$ on a domain $\Omega$ is harmonic if and only if $\bar{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ for all $x\in\Omega$ and all $r$ for which $\overline{B}(\boldsymbol{x},|r|)\subset\Omega$.*

In general, we say a function $u$ satisfies the *mean value property* if $\bar{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ whenever $\overline{B}(\boldsymbol{x},|r|)\subset\Omega$. We hall see below (Theorem 4) that the mean value property characterizes harmonic functions. But first, we collect an easy consequence of (4).[2]

**Proposition 3.** *For any $C^2$ function $u$, we have*

$$\Delta u(\boldsymbol{x}) = n\bar{u}''_{\boldsymbol{x}}(0).$$

*Proof.* The function $\bar{u}_{\boldsymbol{x}}$ is even, so $\bar{u}'_{\boldsymbol{x}}(0) = 0$. Therefore, letting $r\to 0$ in (4), we arrive at the stated result. ∎

**Theorem 4** (The mean value property and regularity)**.** *Assume that a continuous function $u$ satisfies the mean value property on a domain $\Omega$. Then $u$ is infinitely differentiable, and is therefore harmonic. In particular, every harmonic function is infinitely differentiable.*

[2] Faded out because although the result is interesting, we shall not use it.

*Proof.* This proof may seem long, but only because we use it to develop some tools that have wider applicability.

First, define a *standard mollifier* $\rho\colon \mathbb{R}^n \to \mathbb{R}$. Here is one of many possible definitions:

$$\rho(\boldsymbol{x}) = \begin{cases} ae^{1/(|\boldsymbol{x}|^2-1)}, & |\boldsymbol{x}| < 1 \\ 0, & |\boldsymbol{x}| \geq 1, \end{cases}$$

where the constant $a > 0$ is chosen to ensure that

$$\int_{\mathbb{R}^n} \rho \, \mathrm{d}\boldsymbol{x} = 1.$$

That is one of the defining qualities of a standard mollifier. The others are: That $\rho \geq 0$ everywhere, that it vanishes outside the unit ball, that it is infinitely differentiable, and is radially symmetric – that is, a function of $|\boldsymbol{x}|$ alone.

For any $\delta > 0$ we can squeeze the mollifier to fit inside a ball of radius $\delta$:

$$\rho_\delta(\boldsymbol{x}) = \frac{1}{\delta^n}\rho\Big(\frac{\boldsymbol{x}}{\delta}\Big),$$

so that $\rho_\delta$ also has integral 1, but vanishes outside the ball $B(0,\delta)$.

Now we consider the convolution product

$$u * \rho_\delta(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{y})\rho_\delta(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}.$$

This is defined for all $x \in \Omega$ with a distance less than $\delta$ to the complement of $\Omega$. Thus, for any $x \in \Omega$, we can make $\delta$ small enough so that $u * \rho_\delta$ is defined at $\boldsymbol{x}$.

Moreover, $u * \rho_\delta$ is infinitely differentiable: This is proved by differentiating with respect to the components of $\boldsymbol{x}$ under the integral sign, as much as you like.

Finally, the mean value property of $u$ and the radial symmetry of $\rho_\delta$ combine to ensure that $u(\boldsymbol{x}) = u * \rho(\boldsymbol{x})$ for all $\boldsymbol{x}$ where $u * \rho$ is defined.

For a detailed argument, write

$$u * \rho_\delta(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{x}-\boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}$$

and write the integral in polar form:

$$\begin{aligned} u * \rho_\delta(\boldsymbol{x}) &= \int_0^\delta \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{x}-\boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, \mathrm{d}r \\ &= \int_0^\delta \int_{\mathbb{S}^{n-1}} u(\boldsymbol{x}-r\boldsymbol{y})\rho_\delta(r\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, r^{n-1} \, \mathrm{d}r. \end{aligned} \tag{5}$$





Now use the radial symmetry: $\mathring{\rho}_\delta(r) \coloneqq \rho_\delta(r\boldsymbol{y})$ is independent of $y \in \mathbb{S}^{n-1}$, so this factor can be moved outside the inner integral. Next, use the mean value property of $u$:

$$\int_{\mathbb{S}^{n-1}} u(\boldsymbol{x}-r\boldsymbol{y})\rho_\delta(r\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y} = A_n u(\boldsymbol{x})\mathring{\rho}_\delta(r).$$

But $u(\boldsymbol{x})$ is a constant, which we move outside the outer integral in (5). We are left with

$$u * \rho_\delta(\boldsymbol{x}) = u(\boldsymbol{x}) \int_0^\delta r^{n-1} A_n \mathring{\rho}_\delta(r) \, \mathrm{d}r = u(\boldsymbol{x}) \int_{B(\boldsymbol{0},\delta)} \rho_\delta(z) \, \mathrm{d}^n z = u(\boldsymbol{x}).$$

Since $u * \rho_\delta$ is $C^\infty$, then so is $u$, wherever $u * \rho_\delta$ is defined. By making $\delta$ as small as we wish, we conclude that $u \in C^\infty(\Omega)$, as was our goal. ∎

**Corollary 5.** *A uniform limit of a sequence of harmonic functions is harmonic.*

*Proof.* Left as an exercise for the reader. ∎

The gradient bound below will turn out to be useful later on. But first, note that the same method used to prove that $u(\boldsymbol{x}) = u * \rho_\delta(\boldsymbol{x})$ above also shows that $u(\boldsymbol{x})$ is the average of $u$ over a *ball* centred at $\boldsymbol{x}$.

**Proposition 6** (Gradient bound)**.** *Assume that $u$ is a bounded harmonic function on a domain $\Omega \subset \mathbb{R}^n$. Then*

$$|\nabla u(\boldsymbol{x})| \leq \frac{n \sup_\Omega |u|}{\operatorname{dist}(\boldsymbol{x}, \partial\Omega)}$$

*Proof.* Let $\boldsymbol{v}$ be any unit vector. Each component of $\nabla u$ is harmonic, and hence so is $\boldsymbol{v} \cdot \nabla u$. Applying the mean value theorem (ball version) where $\overline{B}_r(\boldsymbol{x}) \subset \Omega$, we get

$$\begin{aligned} \boldsymbol{v} \cdot \nabla u(\boldsymbol{x}) &= \frac{n}{A_n r^n} \int_{B_r(\boldsymbol{x})} \boldsymbol{v} \cdot \nabla u(\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y} \\ &= \frac{n}{A_n r^n} \int_{\partial B_r(\boldsymbol{x})} \boldsymbol{n} \cdot \boldsymbol{v} u(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \quad \text{because } \boldsymbol{v} \cdot \nabla u = \nabla \cdot (\boldsymbol{v} u), \end{aligned}$$

and therefore

$$\big|\boldsymbol{v} \cdot \nabla u(\boldsymbol{x})\big| \leq \frac{n}{A_n r^n} A_n r^{n-1} \sup_\Omega |u| = \frac{n \sup_\Omega |u|}{r}.$$

Now let $r \to \operatorname{dist}(\boldsymbol{x}, \partial\Omega)$ to finish the proof. ∎

## The maximum principle

**Definition.** A $C^2$ function $u$ is called *subharmonic* if $\Delta u \geq 0$, and *superharmonic* if $\Delta u \leq 0$. Thus it is harmonic if and only if it is both subharmonic and superharmonic. (The reason for the naming will become clear later; see Corollary 9.) Clearly, $u$ is superharmonic if and only if $-u$ is subharmonic.

Note that, in the one-dimensional case ($n = 1$), subharmonic functions are *convex*, while superharmonic functions are *concave*.[3] This is a great help to intuition – think of this special case as we explore the properties of sub- and superharmonic functions below!

**Theorem 7** (Strong maximum principle). *Assume that $u \in C^2(\Omega)$ is subharmonic in a region $\Omega \subseteq \mathbb{R}^n$. If $u$ has a global maximum in $\Omega$, then $u$ is constant.*

*Proof.* Let $M$ be the global maximum of $u$, and put

$$S = \{\, \boldsymbol{x} \in \Omega \mid u(\boldsymbol{x}) = M \,\}.$$

Then $S$ is a closed subset of $\Omega$, by the continuity of $u$. It is also nonempty by assumption.

Consider any $\boldsymbol{x} \in S$. From (4) and the subharmonicity of $u$, we get $\bar{u}'_{\boldsymbol{x}}(r) \geq 0$ for $r > 0$. Thus we get $\bar{u}_{\boldsymbol{x}}(r) \geq \bar{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x}) = M$ for $r > 0$ (so long as $\overline{B}(\boldsymbol{x}, r) \subset \Omega$). But $u \leq M$ everywhere, and if $u < M$ anywhere on the sphere $\partial B(\boldsymbol{x}, r)$, we would get $\bar{u}_{\boldsymbol{x}}(r) < M$. Thus $u = M$ in some neighbourhood of $\boldsymbol{x}$. This means that $S$ is open.

We have shown that $S$ is an open, closed, and nonempty subset of the connected set $\Omega$. Therefore $S = \Omega$, and the proof is complete. ∎

**Remark.** Obviously, we obtain a strong *minimum* principle for *superharmonic* functions by multiplying by $-1$. In particular, a non-constant harmonic function cannot attain a minimum or maximum value anywhere in $\Omega$.

**Corollary 8** (Weak maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is subharmonic. Then*

$$\max \{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \overline{\Omega} \,\} = \max \{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \partial\Omega \,\}.$$

*In particular, a harmonic function which is continuous on $\overline{\Omega}$ attains its minimum and maximum values on the boundary $\partial\Omega$.*

*Proof.* The weak principle is an obvious consequence of the strong principle together with the existence of a maximum of the continuous function $u$ on the compact set $\overline{\Omega}$.

[3] Or, as calculus textbooks perversely(?) call it, "concave up" and "concave down", respectively.





However, it is worth noting that a much more elementary proof exists. Namely, for any $\varepsilon > 0$, let $v(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon|\boldsymbol{x}|^2$, and note that then $\Delta v > 0$. But $\Delta v(\boldsymbol{x}) \leq 0$ if $\boldsymbol{x}$ is an interior minimum point for $v$, so $v$ cannot have any maximum in the interior. Thus for any $\boldsymbol{x} \in \Omega$,

$$u(\boldsymbol{x}) = v(\boldsymbol{x}) - \varepsilon|\boldsymbol{x}|^2 \leq \max_{\partial\Omega} v \leq \max_{\partial\Omega} u + \varepsilon \max_{\boldsymbol{x}\in\partial\Omega} |\boldsymbol{x}|^2.$$

Now let $\varepsilon \to 0$ to arrive at the conclusion $u(\boldsymbol{x}) \leq \max_{\partial\Omega} u$. ∎

Our next result explains the terms *sub*- and *super*harmonic: A subharmonic function is below, and a superharmonic above, a harmonic function given the same boundary data.

**Corollary 9.** *Assume that $\Omega$ is a bounded domain, and that $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$, with $u$ harmonic in $\Omega$. If $v$ is subharmonic and $v \leq u$ on $\partial\Omega$, then $v \leq u$ in $\Omega$, while if $v$ is superharmonic and $v \geq u$ on $\partial\Omega$, then $v \geq u$ in $\Omega$.*

*Proof.* Apply the weak maximum principle to $v - u$ if $v$ is subharmonic, or to $u - v$ if $v$ is superharmonic. ∎

**Remark.** Corollary 9 suggests a strategy for proving existence of a solution to the Dirichlet problem for the Laplace equation: Assume we are trying to find a harmonic function on $\Omega$ with the given boundary value $g$. Consider the pointwise *supremum* of all subharmonic functions which are $\leq g$ on $\partial\Omega$, and the pointwise *infimum* of all superharmonic functions which are $\geq g$ on $\partial\Omega$. If the two functions coincide, they might provide a solution to the problem. (The truth isn't quite that rosy, depending on the domain.) This is the basis for *Perron's method*, described later.

## The Poisson equation

We now turn our study to the *Poisson equation*:

$$-\Delta u = f \tag{6}$$

where $f$ is a known continuous function. (It *must* be continuous to allow for classical, i.e., $C^2$, solutions $u$.)

Referring all the way back to (2), we quickly get the following generalization of Proposition 1:

**Proposition 10.** *A $C^2$ function $u$ on a domain $\Omega$ solves the Poisson equation* (6) *if and only if*

$$-\int_{\partial\omega} \partial_n u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_{\omega} f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} \tag{7}$$

*for all bounded domains $\omega$ with $\overline{\omega} \subset \Omega$ having piecewise $C^1$ boundary. It is sufficient to consider balls $\omega = B(\boldsymbol{x}, r)$.*

As an example, we consider a Poisson equation with a radially symmetric right hand side $f(\boldsymbol{x}) = \mathring{f}(|\boldsymbol{x}|)$. We expect to find a radially symmetric solution $u(\boldsymbol{x}) = \mathring{u}(|\boldsymbol{x}|)$. Now (7) with $\omega = B(\boldsymbol{0}, r)$ becomes

$$-A_n r^{n-1}\mathring{u}'(r) = A_n \int_0^r \mathring{f}(s) s^{n-1}\,\mathrm{d}s.$$

Taking the derivative and rearranging turns this into the ODE

$$-\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\big(r^{n-1}\mathring{u}'(r)\big) = \mathring{f}(r). \tag{8}$$

A direct calculation reveals that indeed,

$$\Delta\mathring{u}(|\boldsymbol{x}|) = \mathring{u}''(|\boldsymbol{x}|) + \frac{n-1}{|\boldsymbol{x}|}\mathring{u}'(|\boldsymbol{x}|) = \frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\big(r^{n-1}\mathring{u}'(r)\big)\Big|_{r=|\boldsymbol{x}|}$$

so a solution to (8) will in fact solve the Poisson equation in the radially symmetric case.

The ODE (8) is easily integrated:

$$-r^{n-1}\mathring{u}'(r) = \int_0^r \mathring{f}(s)s^{n-1}\,\mathrm{d}s = \frac{1}{A_n}\int_{B(\boldsymbol{0},r)} f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} =: \frac{m(r)}{A_n}.$$





If we let $f$ approach a Dirac's $\delta$, we would end up with $m(r) = 1$ for all $r$. Thus, after integrating once more, we arrive at the solution $u = \Phi$, where $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$\Phi(\boldsymbol{x}) = \frac{-\ln(|\boldsymbol{x}|)}{2\pi} \quad \text{for } n = 2,$$
$$\Phi(\boldsymbol{x}) = \frac{1}{(n-2)A_n|\boldsymbol{x}|^{n-2}} \quad \text{for } n \geq 3.$$

For $n \geq 3$, this gives the unique radially symmetric solution which vanishes at infinity; for $n = 2$, however, no particular value of the dropped constant of integration distinguishes itself.

Now, put $m = 1$, replace $f(\boldsymbol{x})$ by $\varepsilon^{-n} f(\boldsymbol{x}/\varepsilon)$, and let $\varepsilon \to 0$. The resulting solution $u$ will converge pointwise to $\Phi$ (except at $\boldsymbol{x} = \boldsymbol{0}$), while $f$ becomes a Dirac $\delta$ in the limit. Thus we are tempted to conclude that

$$-\Delta\Phi = \delta$$

This is indeed true, but we first need to get into the theory of distributions in order to understand the rigorous meaning of the above equation.

However, we can make one useful observation. The following lemma says precisely what you would expect, given Proposition 10 and $-\Delta\Phi = \delta$:

**Lemma 11.** *Given a bounded region $\omega \subset \mathbb{R}^n$ with piecewise $C^1$ boundary and $\boldsymbol{0} \notin \partial\omega$,*

$$\int_{\partial\omega} \partial_n\Phi\,\mathrm{d}S = \begin{cases} -1 & \text{if } \boldsymbol{0} \in \omega, \\ 0 & \text{if } \boldsymbol{0} \notin \overline{\omega}. \end{cases}$$

*Proof.* The second case is a consequence of the harmonicity of $\Phi$ and Proposition 1. For the first case, it is easily verified by direct calculation in the case of a ball $\omega = B(\boldsymbol{0}, r)$. In the general case for $\boldsymbol{0} \in \omega$, pick $r > 0$ small enough so that $\overline{B}(\boldsymbol{0}, r) \subset \omega$, and apply the second case to $\omega \setminus B(\boldsymbol{0}, r)$. Thus

$$0 = \int_{\partial(\omega\setminus B(\boldsymbol{0},r))} \partial_n\Phi\,\mathrm{d}S = \int_{\partial\omega} \partial_n\Phi\,\mathrm{d}S - \int_{\partial B(\boldsymbol{0},r)} \partial_n\Phi\,\mathrm{d}S = \int_{\partial\omega} \partial_n\Phi\,\mathrm{d}S + 1,$$

and the proof is complete. (The minus sign in the above calculation is because the unit normal pointing *out* of $\omega \setminus B(\boldsymbol{0}, r)$ at $\partial B(\boldsymbol{0}, r)$ points *into* $B(\boldsymbol{0}, r)$.) ∎

We call $\Phi$ the *fundamental solution* for the Poisson equation. Even lacking the abstract theory, we can use it to solve the general Poisson equation.

**Proposed solution to the Poisson equation:**

$$u(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} \tag{9}$$

We expect this to solve the equation because formally, we get

$$-\Delta u = -\Delta(\Phi * f) = (-\Delta\Phi) * f = \delta * f = f.$$

However, this calculation is hard to justify by elementary means, but we can do it with a bit of help from Proposition 10.

**Theorem 12.** *Assume that $f \in C_c^2(\mathbb{R}^n)$. Then the function $u$ given by* (9) *is a solution to the Poisson equation* (6).

*Proof.* First, the assumptions imply that $u \in C^2$, since $\Phi$ is locally integrable. Let $\omega \subset \mathbb{R}^n$ be a bounded domain with piecewise $C^1$ boundary. Then we calculate:

$$\begin{aligned}
-\int_{\partial\omega} \partial_n u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) &= -\int_{\partial\omega} \partial_n\Bigl(\int_{\mathbb{R}^n} \Phi(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}\Bigr)\,\mathrm{d}S(\boldsymbol{x}) \\
&= -\int_{\mathbb{R}^n} f(\boldsymbol{y})\Bigl(\int_{\partial\omega} \partial_n\Phi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{x})\Bigr)\,\mathrm{d}^n\boldsymbol{y} \\
&= -\int_{\mathbb{R}^n} f(\boldsymbol{y})\Bigl(\int_{\partial(\omega-\boldsymbol{y})} \partial_n\Phi(\boldsymbol{z})\,\mathrm{d}S(\boldsymbol{z})\Bigr)\,\mathrm{d}^n\boldsymbol{y} \\
&\overset{*}{=} \int_{\mathbb{R}^n} f(\boldsymbol{y})[\boldsymbol{0} \in \omega - \boldsymbol{y}]\,\mathrm{d}^n\boldsymbol{y} = \int_{\mathbb{R}^n} f(\boldsymbol{y})[\boldsymbol{y} \in \omega]\,\mathrm{d}^n\boldsymbol{y} \\
&= \int_{\omega} f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}.
\end{aligned}$$

In the above calculation, $\omega - \boldsymbol{y} = \{\boldsymbol{x}-\boldsymbol{y} \mid \boldsymbol{x} \in \omega\}$, and the square brackets are *Iverson brackets*: For any statement $S$,

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true,} \\ 0 & \text{if } S \text{ is false.} \end{cases}$$

We used Lemma 11 in the equality marked with an asterisk. By Proposition 10, the proof is now complete. ∎





**A more conventional proof of Theorem 12**

Our main tool is Green's second identity. We often need to employ it on a region containing the singularity of $\Phi$, which is of course not allowed. So the trick is to remove a ball centered on the singularity, use Green's second identity on the rest of the region, and then let the radius of the ball tend to zero. This leads to two limits handled by the following two lemmas.

**Lemma 13.** *If the function $v$ is continuous near $\boldsymbol{0}$, then*

$$-\int_{\partial B(\boldsymbol{0},r)} v\,\partial_n\Phi\,\mathrm{d}S = \fint_{\partial B(\boldsymbol{0},r)} v\,\mathrm{d}S \to v(\boldsymbol{0}) \qquad \text{as } r \to 0.$$

*Proof.* The identity follows from the fact that the normal derivative on $\partial B(\boldsymbol{0},r)$ is the radial derivative, with the constant value $\partial_n\Phi = -1/(A_n r^{n-1})$ on the sphere. And then the limit follows by continuity of $v$. ∎

**Lemma 14.** *If the function $v$ is $C^1$ near $\boldsymbol{0}$, then*

$$\int_{\partial B(\boldsymbol{0},r)} \Phi\,\partial_n v\,\mathrm{d}S \to 0 \qquad \text{as } r \to 0.$$

*Proof.* The normal derivative $\partial_n v$ is really $\boldsymbol{n} \cdot \nabla v$, but $\boldsymbol{n}$ is a unit vector, and $\nabla v$ is bounded near $\boldsymbol{0}$, since $v$ is assumed to be $C^1$. Because the integral of $|\Phi|$ tends to zero as $r \to 0$, the integral of the first term will vanish in the limit. ∎

**Theorem 15.** *Assume that $f \in C_c^2(\mathbb{R}^n)$. Then the function $u$ given by* (9) *is a solution to the Poisson equation* (6).

*Proof.* Thanks to translation invariance, we only need to prove that $-\Delta(\Phi * f)(\boldsymbol{0}) = f(\boldsymbol{0})$. (For any $\boldsymbol{x}_0 \in \mathbb{R}^n$, put $\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}_0 + \boldsymbol{x})$. Then $\Phi * \tilde{f}(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}_0 + \boldsymbol{x})$, so $-\Delta(\Phi * \tilde{f})(\boldsymbol{0}) = \tilde{f}(\boldsymbol{0})$ implies $-\Delta(\Phi * f)(\boldsymbol{x}) = f(\boldsymbol{x})$.)

First, we note that

$$\int_{\partial B(\boldsymbol{0},r)} \Phi\,\mathrm{d}^n\boldsymbol{x} = \begin{cases} -r\ln r & \text{for } n = 2, \\ r/(n-2) & \text{for } n \geq 3. \end{cases}$$

Integrating this with respect to $r$, we conclude that $\Phi$ is integrable (meaning the integral of $|\Phi|$ is finite) over $B(\boldsymbol{0},r)$, and hence over any bounded subset of $\mathbb{R}^n$. From this, we conclude that not only is

$$u(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y}) f(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}$$

well defined, but $u$ is $C^2$ as well, and in fact

$$\Delta u = \Phi * \Delta f.$$

This is easy if $f$ has compact support; it is also true if $f$ and its derivatives up to second order vanish sufficiently fast at infinity, but we are not going to bother with this refinement.

In particular,

$$\Delta u(\mathbf{0}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y})\,\Delta f(-\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y})\,\Delta f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}$$

(using the symmetry of $\Phi$ to get rid of the minus sign for convenience). We want to use Green's second identity to move the Laplacian to $\Phi$ instead. But then we need to restrict attention to a bounded region, and we need to avoid the singularity of $\Phi$ at the origin. (It is this singularity that allows the answer to be non-zero, after all.) Pick $R$ sufficiently large so $f(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \geq R$, so that integrating over $B(\mathbf{0}, R)$ instead of $\mathbb{R}^n$ in the integral above does not change its value. Then, thanks to the integrabilty of $\Phi$ near the origin, we find

$$\Delta u(\mathbf{0}) = \lim_{r\to 0} \int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)} \Phi(\boldsymbol{y})\,\Delta f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}.$$

This integral can be transformed by Green's second identity, resulting in

$$\Delta u(\mathbf{0}) = \lim_{r\to 0}\Bigl(\int_{B(\mathbf{0},R)\setminus B(\mathbf{0},r)} \Delta\Phi(\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} + \int_{\partial(B(\mathbf{0},R)\setminus B(\mathbf{0},r))} (\Phi\partial_n f - f\partial_n\Phi)\,\mathrm{d}S\Bigr).$$

The first integral vanishes because $\Delta\Phi = 0$, and in the second integral we can ignore the outer boundary $\partial B(\mathbf{0}, R)$ because $f = \partial_n f = 0$ there. Finally, the normal vector $\boldsymbol{n}$ points *inward* on the inner boundary $\partial B(\mathbf{0}, r)$, so we get a sign change when we consider the outward point normal instead. Thus we have

$$\Delta u(\mathbf{0}) = \lim_{r\to 0} \int_{\partial B(\mathbf{0},r)} (f\partial_n\Phi - \Phi\partial_n f)\,\mathrm{d}S.$$

The two preceding lemmas then yield $\Delta u(\mathbf{0}) = -f(\mathbf{0})$. ∎





**Bounded domains and Green's function.** Let $\Omega$ be a bounded domain with piecewise $C^1$ boundary. The weak maximum principle (even without the "nice" boundary) immediately shows that the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{DP}$$

has at most one solution for any given data $f$ and $g$.

Our goal in this section is to generalize the representation formula (9) to this setting. We first concentrate on homogeneous boundary data, i.e., $g = 0$. It will turn out that we get the case for $g \neq 0$ "for free".

First, rewrite (9) to read

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}.$$

A similar formula for solutions on $\Omega$ might look like

$$u(\boldsymbol{x}) = \int_{\Omega} G_{\boldsymbol{y}}(\boldsymbol{x}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}$$

instead. To ensure that this satisfies $u = 0$ on $\partial\Omega$, we want $G_{\boldsymbol{y}}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial\Omega$. Apart from that, we want $G_{\boldsymbol{y}}(\boldsymbol{x})$ (as a function of $\boldsymbol{x}$) to be as much "like" $\Phi(\boldsymbol{x}-\boldsymbol{y})$ as possible. To make that precise:

**Definition.** Assume that, for each $\boldsymbol{y} \in \Omega$, there exists a function $H_{\boldsymbol{y}} \in C^2(\overline{\Omega})$ which is harmonic in $\Omega$ and satisfies

$$H_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y}) \quad \text{for } \boldsymbol{x} \in \partial\Omega.$$

By the maximum principle, this function is unique. Then the function $G_{\boldsymbol{y}}$ defined by

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y}) - H_{\boldsymbol{y}}(\boldsymbol{x})$$

is called the *Green's function* associated with $\Omega$. By construction, this function is $C^2$ on $\overline{\Omega}\setminus\{\boldsymbol{y}\}$ and harmonic on $\Omega\setminus\{\boldsymbol{y}\}$, and it vanishes on $\partial\Omega$.

To discover how this helps us write a solution formula for the Dirichlet problem, we concentrate first on the $\Phi$ part.

For simplicity, assume that $\boldsymbol{y} = \mathbf{0}$. If $\varepsilon > 0$ is small enough, $\overline{B}(\mathbf{0}, \varepsilon) \subset \Omega$. Put $\Omega_\varepsilon = \Omega \setminus \overline{B}(\mathbf{0}, \varepsilon)$, and apply Green's second identity to $\Phi$ and an arbitrary function $u \in C^2(\overline{\Omega})$:

$$\begin{aligned}\int_{\Omega_\varepsilon} \bigl(u\,\Delta\Phi - \Phi\,\Delta u\bigr)\,\mathrm{d}^n\boldsymbol{x} &= \int_{\partial\Omega_\varepsilon} \bigl(u\,\partial_n\Phi - \Phi\,\partial_n u\bigr)\,\mathrm{d}S\\ &= \int_{\partial\Omega} \bigl(u\,\partial_n\Phi - \Phi\,\partial_n u\bigr)\,\mathrm{d}S - \int_{\partial B(\mathbf{0},\varepsilon)} \bigl(u\,\partial_n\Phi - \Phi\,\partial_n u\bigr)\,\mathrm{d}S,\end{aligned}$$

where the minus sign in front of the last integral is due to the direction of the normal vector $\boldsymbol{n}$ on $\partial B(\mathbf{0},\varepsilon)$ pointing out of the ball, whereas the normal vector on that part of $\partial\Omega_\varepsilon$ points into the ball.

Now let $\varepsilon \to 0$. The red term on the left is already zero, while the boundary integral of the red term on the right will vanish in the limit by Lemma 14. Finally, the integral of the green term will converge to $-u(\mathbf{0})$ by Lemma 13, so we end up with the representation formula

$$u(\boldsymbol{y}) = -\int_\Omega \Phi\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega}\big(u\,\partial_n\Phi - \Phi\,\partial_n u\big)\,\mathrm{d}S,\qquad \boldsymbol{y}\in\Omega. \tag{10}$$

(We proved it only for $\boldsymbol{y}=\mathbf{0}$, but translation invariance ensures that the result generalizes. Or you could rerun the proof with $\Phi(\boldsymbol{x})$ replaced by $\Phi_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y})$ and $B(\mathbf{0},\varepsilon)$ by $B(\boldsymbol{y},\varepsilon)$.)

Repeat the same calculation with $\Phi$ replaced by $H_{\boldsymbol{y}}$. This is much easier, since $H_{\boldsymbol{y}}$ is harmonic we do not need to remove a ball: Green's second identity immediately yields

$$\int_\Omega\big(u\,\Delta H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\,\Delta u\big)\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\Omega}\big(u\,\partial_n H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\partial_n u\big)\,\mathrm{d}S,$$

where again the red term vanishes. Rearranging this into

$$\int_\Omega H_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} + \int_{\partial\Omega}\big(u\,\partial_n H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\partial_n u\big)\,\mathrm{d}S = 0$$

and adding it to the right hand side of (10), we obtain

$$u(\boldsymbol{y}) = -\int_\Omega G_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega}\big(u\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n u\big)\,\mathrm{d}S$$

where the red term is zero by construction, so we finally have

$$u(\boldsymbol{y}) = -\int_\Omega G_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega} u\,\partial_n G_{\boldsymbol{y}}\,\mathrm{d}S,\qquad \boldsymbol{y}\in\Omega.$$

We have proved

**Theorem 16.** *If $\Omega\subset\mathbb{R}^n$ is a bounded region with $C^1$ boundary, and if $\Omega$ admits a Green's function $G$, then the solution $u\in C^2(\overline{\Omega})$ of* (DP), ***if it exists***, *is given by*

$$u(\boldsymbol{y}) = \int_\Omega G_{\boldsymbol{y}}(\boldsymbol{x})\,f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega}\partial_n G_{\boldsymbol{y}}(\boldsymbol{x})\,g(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}),\qquad \boldsymbol{y}\in\Omega. \tag{11}$$

Notice the careful wording: We have not shown that the Dirichlet problem has a solution. We have only shown what it must be, if it solves the problem.

We now show that Green's function is symmetric.





**Proposition 17.** *Assume that $\Omega$ is a bounded region with piecewise $C^1$ boundary, and which admits a Green's function. Then the Green's function $G$ for this region satisfies $G_{\boldsymbol{y}}(\boldsymbol{x}) = G_{\boldsymbol{x}}(\boldsymbol{y})$, whenever $\boldsymbol{x},\boldsymbol{y}\in\Omega$.*

*Proof.* First a "physicists' proof" to reveal the essentials: Since $G_{\boldsymbol{x}}$ vanishes on $\partial\Omega$ and $-\Delta G_{\boldsymbol{x}}(\boldsymbol{z}) = \delta(\boldsymbol{z}-\boldsymbol{x})$, and similarly for $G_{\boldsymbol{y}}$, Green´s second identity yields

$$\begin{aligned}0 &= \int_{\partial\Omega}\big(G_{\boldsymbol{x}}\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n G_{\boldsymbol{x}}\big)\,\mathrm{d}S = \int_\Omega\big(G_{\boldsymbol{x}}\,\Delta G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\Delta G_{\boldsymbol{x}}\big)\,\mathrm{d}^n\boldsymbol{z}\\ &= \int_\Omega\big(-G_{\boldsymbol{x}}(\boldsymbol{z})\delta(\boldsymbol{z}-\boldsymbol{y}) + G_{\boldsymbol{y}}(\boldsymbol{z})\delta(\boldsymbol{z}-\boldsymbol{x})\big)\,\mathrm{d}^n\boldsymbol{z} = -G_{\boldsymbol{x}}(\boldsymbol{y}) + G_{\boldsymbol{y}}(\boldsymbol{x}).\end{aligned}$$

Our problem with this proof is that Green's second identity, as we know it, requires $C^2$ functions, which is not the case here. So we remove some small balls around the troublesome points $\boldsymbol{x}$ and $\boldsymbol{y}$: Put $\Omega_\varepsilon = \Omega\setminus\big(\overline{B}(\boldsymbol{x},\varepsilon)\cup\overline{B}(\boldsymbol{y},\varepsilon)\big)$, where $\varepsilon>0$ is small enough so that the two removed balls lie within $\Omega$ and have no point in common.

Now $\Delta G_{\boldsymbol{x}}$ and $\Delta G_{\boldsymbol{y}}$ are harmonic in $\Omega_\varepsilon$, so Green's theorem is applicable:

$$\begin{aligned}0 &= \int_{\Omega_\varepsilon}\big(G_{\boldsymbol{x}}\,\Delta G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\Delta G_{\boldsymbol{x}}\big)\,\mathrm{d}^n\boldsymbol{z} = \int_{\partial\Omega_\varepsilon}\big(G_{\boldsymbol{x}}\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n G_{\boldsymbol{x}}\big)\,\mathrm{d}S\\ &= -\Big(\int_{\partial B(\boldsymbol{x},\varepsilon)} + \int_{\partial B(\boldsymbol{y},\varepsilon)}\Big)\big(G_{\boldsymbol{x}}\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n G_{\boldsymbol{x}}\big)\,\mathrm{d}S,\end{aligned}$$

where we dropped the integral over $\partial\Omega$, because the integrand is zero there. The minus sign is due to the reversed direction of the normal vector on the boundary of the balls versus the boundary of $\Omega_\varepsilon$. We have already dealt with this sort of boundary integral in the proof of Theorem 16, so here we go again! Look at the first of the two boundary integrals above:

$$\int_{\partial B(\boldsymbol{x},\varepsilon)}\big(G_{\boldsymbol{x}}\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n G_{\boldsymbol{x}}\big)\,\mathrm{d}S.$$

The integral of the red term is zero, for $G_{\boldsymbol{x}}$ is constant on $\partial B(\boldsymbol{x},\varepsilon)$, and

$$\int_{\partial B(\boldsymbol{x},\varepsilon)}\partial_n G_{\boldsymbol{y}}\,\mathrm{d}S = \int_{B(\boldsymbol{x},\varepsilon)}\Delta G_{\boldsymbol{y}}\,\mathrm{d}^n\boldsymbol{z} = 0$$

by the divergence theorem. Even without harmonicity, this would vanish in the limit as $\varepsilon\to 0$.

For the green term, note that $-\partial_{\boldsymbol{y}} G_{\boldsymbol{x}} = 1/(A_n\varepsilon^{n-1})$ on $\partial B(\boldsymbol{x},\varepsilon)$, so together with the above calculation we get

$$\int_{\partial B(\boldsymbol{x},\varepsilon)}\big(G_{\boldsymbol{x}}\,\partial_n G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_n G_{\boldsymbol{x}}\big)\,\mathrm{d}S = \fint_{\partial B(\boldsymbol{x},\varepsilon)} G_{\boldsymbol{y}}\,\mathrm{d}S \to G_{\boldsymbol{y}}(\boldsymbol{x})\quad\text{as } \varepsilon\to 0.$$

Similarly,

$$\int_{\partial B(y,\varepsilon)} (G_x\,\partial_n G_y - G_y\,\partial_n G_x)\,\mathrm{d}S = \fint_{\partial B(y,\varepsilon)} G_x\,\mathrm{d}S \to G_x(y) \quad \text{as } \varepsilon \to 0,$$

and the proof is complete. ∎

**Remark.** A different take on the proof of symmetry of Green's function:

Let $f, g \in C(\overline{\Omega})$, and let $u, v$ solve $-\Delta u = f$ and $-\Delta v = g$ in $\Omega$, $u = v = 0$ on $\partial\Omega$. Thus

$$u(y) = \int_\Omega G_y(x)\,f(x)\,\mathrm{d}^n x, \quad v(y) = \int_\Omega G_y(x)\,g(x)\,\mathrm{d}^n x.$$

Since $u$ and $v$ vanish on $\partial\Omega$, Green's second identity applied to $u$ and $v$ implies

$$\int_\Omega u(y)g(y)\,\mathrm{d}^n y = \int_\Omega v(y)f(y)\,\mathrm{d}^n y.$$

Plugging in the above formulas for $u$ and $v$, swapping the integration variables $x$ and $y$ in one of the integrals, and rearranging, we get

$$\int_\Omega\int_\Omega \big(G_y(x) - G_x(y)\big)f(x)g(y)\,\mathrm{d}^n x\,\mathrm{d}^n y = 0,$$

from which the desired conclusion quickly follows. (Take $f$ and $g$ to be approximate delta functions concentrated near two different points in $\Omega$.)

It may be interesting to study the above argument from an elementary linear algebra point of view. The last part resembles the proof that, if $A$ is a quadratic matrix satisfying $y^T Ax = x^t Ay$ for all vectors $x$ and $y$, then $A$ is symmetric. (Just take $x = e_i$ and $y = e_j$.) The first part resembles the proof that the inverse of an invertible symmetric matrix is symmetric. Think of the "matrix" $G_x(y)$ (where $x$ and $y$ play the rôles of matrix indices) as the inverse of the operator $-\Delta$. And the symmetry of this operator is just the identity $\int_\Omega u\,\Delta v = \int_\Omega v\,\Delta u$. (Green's second identity, when both functions vanish on the boundary.)

**Green's function for balls.** Here we compute Green's function for the open unit ball $\mathbb{B}^n = B(\mathbf{0}, 1)$ in $\mathbb{R}^n$. I will drop some exponents and write $\mathbb{B}$ instead of $\mathbb{B}^n$ and $\mathbb{S}$ instead of $\mathbb{S}^{n-1}$ for the unit sphere $\partial\mathbb{B}$.

If $y \in \mathbb{B}$, we need to find a harmonic function on $\mathbb{B}$ with the same values as $\Phi(x - y)$ for $x \in \mathbb{S}$. The trick is to "put a charge" at a suitable point $w$ outside $\mathbb{B}$. It turns out that $w = y/|y|^2$ does the trick. Since $|x| = 1$, we find

$$|x - y|^2 = 1 + |y|^2 - 2x\cdot y$$

and

$$|x - w|^2 = 1 + |w|^2 - 2x\cdot w = 1 + \frac{1}{|y|^2} - 2\frac{x\cdot y}{|y|^2} = \frac{|y - x|^2}{|y|^2},$$





so that

$$|w - x| = \frac{|y - x|}{|y|}.$$

For $n \ge 3$, this gives

$$\Phi(w - x) = \frac{1}{(n-2)A_n|w - x|^{n-2}} = |y|^{n-2}\Phi(y - x),$$

so we should put

$$H_y(x) = \frac{\Phi(w - x)}{|y|^{n-2}} = \frac{1}{(n-2)A_n|y|^{n-2}|w - x|^{n-2}}$$

where we note that

$$|y|^2|w - x|^2 = |y|^2(|w|^2 + |x|^2 - 2x\cdot w) = 1 + |x|^2|y|^2 - 2x\cdot y,$$

so we get

$$G_y(x) = \frac{1}{(n-2)A_n}\left(\frac{1}{|x - y|^{n-2}} - \frac{1}{(1 + |x|^2|y|^2 - 2x\cdot y)^{(n-2)/2}}\right) \quad (n \ge 3)$$

which is indeed symmetric in $x$, $y$ as expected. To use the solution formula (11), we need to compute $-\partial_n G_y(x)$ for $y \in \mathbb{B}$ and $x \in \mathbb{S}$, i.e., for $|y| < 1$ and $|x| = 1$. We find

$$-\partial_n G_y(x) = -\partial_r G_y(rx)\Big|_{r=1} = P_y(x)$$

where $P_y(x)$ is the ($n$-dimensional) *Poisson kernel*

$$P_y(x) = \frac{1 - |y|^2}{A_n\,|x - y|^n}$$

Summarising what we have found, any continuous function on $\overline{\mathbb{B}}$ satisfying $-\Delta u = f$ in $\mathbb{B}$ and $u = g$ on $\partial\mathbb{B}$ is given by (see (11))

$$u(y) = \int_\Omega G_y(x)\,f(x)\,\mathrm{d}^n x + \int_{\partial\Omega} P_y(x)\,g(x)\,\mathrm{d}S(x), \qquad y \in \Omega.$$

The special case where $f = 0$ is known as *Poisson's integral formula* for harmonic functions on the ball.

Next, the same computation for $n = 2$: We find (when $\boldsymbol{x} \in \mathbb{S}$)

$$\Phi(\boldsymbol{w} - \boldsymbol{x}) = -\frac{\ln|\boldsymbol{w} - \boldsymbol{x}|}{2\pi} = \frac{\ln|\boldsymbol{y}|}{2\pi} + \Phi(\boldsymbol{y} - \boldsymbol{x}),$$

so we put

$$H_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln|\boldsymbol{y}|}{2\pi} + \Phi(\boldsymbol{w} - \boldsymbol{x}) = -\frac{\ln(|\boldsymbol{y}||\boldsymbol{w} - \boldsymbol{x}|)}{2\pi}$$

and get

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln|\boldsymbol{y} - \boldsymbol{x}|}{2\pi} + \frac{\ln(1 + |\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y})}{4\pi} \qquad (n = 2)$$

which is again symmetric. We can also write this on the form

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln(|\boldsymbol{x}|^2 + |\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y})}{4\pi} + \frac{\ln(1 + |\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y})}{4\pi} \qquad (n = 2)$$

When solving the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$, we need the normal derivative $\partial_n G_{\boldsymbol{y}}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{S}$ which is simply the derivative of $G_{\boldsymbol{y}}(t\boldsymbol{x})$ taken at $t = 1$, resulting in

$$\partial_n G_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{-2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + 2|\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}}{4\pi(1 + |\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y})} = \frac{|\boldsymbol{y}|^2 - 1}{2\pi(1 + |\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y})}.$$

If we put $r = |\boldsymbol{y}|$ and let $\theta$ be the angle between the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we get

$$-\partial_n G_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

which is the 2-dimensional *Poisson kernel*. Referring back to (11), the solution to the Dirichlet problem $-\Delta u = 0$ on $\mathbb{D}$ with $u(\cos\theta, \sin\theta) = g(\theta)$ should then be given by

$$u(r\cos\varphi, r\sin\varphi) = \frac{1 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{g(\varphi - \theta)}{1 + r^2 - 2r\cos\theta}\,d\theta$$

This is again known as *Poisson's integral formula*. It is essentially the same as the higher-dimensional case, but looks different because of the special structure of the unit circle: We have taken advantage of the ability to add and subtract angles. In





fancier language, the unit circle is a group, whereas higher dimensional spheres are not (except for $\mathbb{S}^3$, the unit quaternions).

It was originally derived using separation of variables and a Fourier analysis. For this, we may note that the Laplace operator in polar coordinates takes the form

$$\Delta u = r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta}.$$

We look for harmonic functions of the form $u = R(r)\Theta(\theta)$, resulting in $r^{-1}(rR')'\Theta + r^{-2}R\Theta'' = 0$. After separating the variables, we are left with $\Theta'' = -\lambda\Theta$ and $(rR')' = \lambda r^{-1}R$. Since $\Theta$ must be $2\pi$-periodic, we must put $\lambda = n^2$ for an integer $n$, so the $R$ equation is $r(rR')' = n^2R$. We try $R = r^k$, and get $rR' = kr^k$, $r(rR')' = k^2r^k$, with the non-trivial solutions $k = \pm n$. Rejecting solutions with negative $k$ (which have a singularity at $r = 0$), we are therefore left with $R = r^{|n|}$. Thus we are led to look for solutions of the form

$$u(r\cos\varphi, r\sin\varphi) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\varphi}.$$

Matching this to $g(\theta)$ at $r = 1$, we should have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta}\,d\theta,$$

and therefore

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} g(\theta) r^{|n|} e^{in(\varphi-\theta)}\,d\theta. = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) P(r, \varphi - \theta)\,d\theta,$$

where

$$P(r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Here we note that

$$\sum_{k=0}^{\infty} r^k e^{\pm ik\theta} = \frac{1}{1 - re^{\pm ik\theta}},$$

so

$$\begin{aligned} P(r, \theta) &= \frac{1}{2\pi}\Big(\frac{1}{1 - re^{ik\theta}} + \frac{1}{1 - re^{-ik\theta}} - 1\Big) \\ &= \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}, \end{aligned}$$

which we recognize as the Poisson kernel introduced above.

**Theorem 18** (Harnack's inequality). *If $u \geq 0$ is continuous on $\overline{\mathbb{B}}(\mathbf{0}, R)$ and harmonic in $B(\mathbf{0}, R) \subset \mathbb{R}^n$, then*

$$\frac{R - |\boldsymbol{x}|}{(R + |\boldsymbol{x}|)^{n-1}} R^{n-2} u(\mathbf{0}) \leq u(\boldsymbol{x}) \leq \frac{R + |\boldsymbol{x}|}{(R - |\boldsymbol{x}|)^{n-1}} R^{n-2} u(\mathbf{0})$$

*when $|x| < R$.*

*Proof.* If $R = 1$, use the inequalities $1 - |\boldsymbol{x}| < |\boldsymbol{x} - \boldsymbol{y}| < 1 + |\boldsymbol{x}|$ (when $|y| = 1$) in Poisson's formula. The case $R \neq 1$ can be reduced to the case $R = 1$ by rescaling. The details are left to the reader. ∎

**The Dirichlet problem on a ball.** The boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

is known as the *Dirichlet problem* for the region $\Omega$.

Simple properties of the Poisson kernel show that the Dirichlet problem for $\mathbb{B}^n$ ($n \geq 2$) has a solution for all $g \in C(\partial\mathbb{B})$.

These properties, easily verified, are:

- $P_{\boldsymbol{y}}(\boldsymbol{x})$ is a harmonic function of $\boldsymbol{y}$ for any $\boldsymbol{x} \in \mathbb{S}$,
- $P_{\boldsymbol{y}}(\boldsymbol{x}) > 0$ for all $\boldsymbol{y} \in \mathbb{B}$ and $\boldsymbol{x} \in \mathbb{S}$,
- $\displaystyle\int_{\mathbb{S}} P_{\boldsymbol{y}}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = 1$, and
- $\displaystyle\lim_{\mathbb{B} \ni \boldsymbol{y} \to \boldsymbol{z}} \int_{\mathbb{S}} P_{\boldsymbol{y}}(\boldsymbol{x}) \big[|\boldsymbol{x} - \boldsymbol{z}| > \delta\big]\,\mathrm{d}S(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}, \boldsymbol{z} \in \mathbb{S}$ and $\delta > 0$.

The claim is now that for any $g \in C(\mathbb{S})$, the function

$$u(\boldsymbol{y}) = \int_{\mathbb{S}} P_{\boldsymbol{y}}(\boldsymbol{x}) g(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})$$

solves the Dirichlet problem. It is clearly harmonic. Further, for any $\boldsymbol{z} \in \mathbb{S}$, we find

$$u(\boldsymbol{y}) - g(\boldsymbol{z}) = \int_{\mathbb{S}} P_{\boldsymbol{y}}(\boldsymbol{x}) \big(g(\boldsymbol{x}) - g(\boldsymbol{z})\big)\,\mathrm{d}S(\boldsymbol{x}),$$





and so

$$\big|u(\boldsymbol{y}) - g(\boldsymbol{z})\big| \leq \int_{\mathbb{S}} P_{\boldsymbol{y}}(\boldsymbol{x}) \big|g(\boldsymbol{x}) - g(\boldsymbol{z})\big|\,\mathrm{d}S(\boldsymbol{x}).$$

Given $\varepsilon > 0$, pick $\delta > 0$ so that $|\boldsymbol{x} - \boldsymbol{z}| < \delta$ implies $|g(\boldsymbol{x}) - g(\boldsymbol{z})| < \varepsilon$. Divide the integral above in two pieces, one where $|\boldsymbol{x} - \boldsymbol{z}| < \delta$ and the rest, and conclude that the integral can be made smaller than $2\varepsilon$ by letting $\boldsymbol{y}$ be sufficiently close to $\boldsymbol{z}$.

## The Dirichlet problem on bounded domains.

Before we get to the meat of the matter, we need to develop some tools. First, we generalise our definitions of sub- and superharmonic functions:

**Definition.** A continuous function $f$ on a domain $\Omega$ is called *subharmonic* if for each $\boldsymbol{x} \in \Omega$

$$f(\boldsymbol{x}) \leq ⨍_{\partial B(\boldsymbol{x}, r)} f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) \tag{12}$$

holds for all sufficiently small $r > 0$. It is called *superharmonic* if the opposite inequality holds.

These notions extend to lower semicontinuous functions for the subharmonic case, and upper semicontinous functions for the superharmonic case. However, we shall have no need for this refinement.

**Lemma 19.** *The weak and strong maximum principles hold for any subharmonic function. When $f$ is subharmonic, the inequality (12) holds whenever $\overline{B}(\boldsymbol{x}, r) \subset \Omega$.*

*Proof.* Inspect the proof of Theorem 7. The only property required of the proof is (12), written in that proof as $\bar{f}_{\boldsymbol{x}}(r) \geq f(\boldsymbol{x})$, for all $\boldsymbol{x}$ and sufficiently small $r$ (but *how* small is allowed to depend on $\boldsymbol{x}$). Therefore, the maximum principle holds.

To show that the defining inequality holds for all $r$, so long as $\overline{B}(\boldsymbol{x}, r) \subset \Omega$, let $u \in C(\overline{B}(\boldsymbol{x}, r))$ be harmonic in $B(\boldsymbol{x}, r)$ and equal to $f$ on $\partial B(\boldsymbol{x}, r)$. The maximum principle applied to the subharmonic function $f - u$ implies $f - u \leq 0$, so $f(\boldsymbol{x}) \leq u(\boldsymbol{x}) = ⨍_{B(\boldsymbol{x}, r)} f\,\mathrm{d}S$. ∎

We shall write $f_1 \vee f_2$ and $f_1 \wedge f_2$ for the *pointwise maximum* and *pointwise minimum*, respectively, of two functions $f_1$, $f_2$:

$$(f_1 \vee f_2)(\boldsymbol{x}) = \max\big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\big), \quad (f_1 \wedge f_2)(\boldsymbol{x}) = \min\big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\big).$$

**Proposition 20.** *If $f_1$ and $f_2$ are subharmonic, then so is $f_1 \vee f_2$.*
*If $f_1$ and $f_2$ are superharmonic, then so is $f_1 \wedge f_2$.*

*Proof.* To prove the first statement, note that

$$f_1(\boldsymbol{x}) \le \fint_{\partial B(\boldsymbol{x},r)} f_1(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) \le \fint_{\partial B(\boldsymbol{x},r)} (f_1 \vee f_2)(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

since $f_1$ is subharmonic and $f_1 \le f_1 \vee f_2$. The same result holds with $f_1$ and $f_2$ interchanged, and the result follows.

The second statement follows from the first by multiplying by $-1$. ∎

**Definition.** If $f \in C(\Omega)$ is subharmonic and $B$ is an open ball with $\overline{B} \subset \Omega$, the *harmonic lifting* of $f$ on $B$ is the function $\tilde{f} \in C[\Omega]$ which equals $f$ outside $B$ and is harmonic inside $B$. (In other words, it solves the Dirichlet problem in $B$ with boundary values $f\big|_{\partial\Omega}$.)

**Proposition 21.** *The harmonic lifting $\tilde{f}$ of a subharmonic function $f$ is itself subharmonic, and $\tilde{f} \ge f$.*

*Proof.* The inequality follows inside $B$ by applying the maximum principle to $f - \tilde{f}$ (which is subharmonic in $B$, and zero on $\partial B$). Outside $B$, it is an equality.

To show that $\tilde{f}$ is subharmonic, consider any $\boldsymbol{x} \in \Omega$. If $\boldsymbol{x} \notin B$, then

$$\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}) \le \fint_{\partial B(\boldsymbol{x},r)} f(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) \le \fint_{\partial B(\boldsymbol{x},r)} \tilde{f}(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}).$$

On the other hand, if $\boldsymbol{x} \in B$, then $B(\boldsymbol{x}, r) \subset B$ for any sufficiently small $r > 0$. Since $\tilde{f}$ is harmonic in $B$,

$$\tilde{f}(\boldsymbol{x}) = \fint_{\partial B(\boldsymbol{x},r)} \tilde{f}(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

and the proof is complete. ∎

It is now time to present our candidate for a solution to the Dirichlet problem on a general bounded region.

**Theorem 22.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded region, and let $g \in C(\partial\Omega)$. Call a function $f \in C(\overline{\Omega})$ a* subsolution *if $f$ is subharmonic in $\Omega$ and $f \le g$ on $\partial\Omega$, and define $u\colon\ \Omega \to \mathbb{R}$ by*

$$u(\boldsymbol{x}) = \sup\{\, f(\boldsymbol{x}) \mid f \text{ is a subsolution} \,\}.$$

*Then $u$ is harmonic in $\Omega$.*

*Proof.* Fix an open ball $B$ with $\overline{B} \subset \Omega$. We shall show that $u$ is harmonic on $B$. Since this will hold for any such $B$, it follows that $u$ is harmonic in $\Omega$.

We need a bigger ball $B^*$ in addition, with $\overline{B} \subset B^*$ and $\overline{B^*} \subset \Omega$. For any subsolution $f$, define $\tilde{f}$ to be the harmonic lifting of $f$ on $B$.





Write $M = \max_{\partial\Omega} |g|$. The maximum principle shows that $f \le M$ for any subsolution $f$. Also, since the constant function $-M$ is a subsolution, $(-M) \vee f$ is a (larger) subsolution, so we may as well only consider subsolutions $f \ge -M$, or put more simply, $|f| \le M$.

Whenever $f$ is a subsolution with $|f| \le M$, so is $\tilde{f}$. Moreover $\tilde{f}$ is harmonic on $B^*$. It now follows from Proposition 6 that $|\nabla \tilde{f}| \le L$ in $B$ for some constant $L$ (only dependent on $B$, $B^*$ and $M$). From this, we get the estimate $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le L|\boldsymbol{x} - \boldsymbol{y}|$ for $\boldsymbol{x}, \boldsymbol{y} \in B$.

Now let the sequence $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ form a dense subset of $B$. For each $k \in \mathbb{N}$, let $f_k$ be a subsolution with $|f_k| \le M$ and $f_k(x_k) > u(x_k) - 1/k$. Put $F_k = f_1 \vee f_2 \vee \dots \vee f_k$. This is again a subsolution, and so is $\tilde{F}_k$, with $|\tilde{F}_k| \le M$ and $\tilde{F}_k(\boldsymbol{x}_j) > u(x_j) - 1/j$ for $1 \le j \le k$. We claim that $\tilde{F}_k \to u$ uniformly on $B$. Since each function $\tilde{F}_k$ is harmonic on $B$, and hence satisfies the mean value property (MVP) on $B$, the uniform limit $u$ will also satisfy MVP, and hence be harmonic.

To prove the claim, let $\varepsilon > 0$. Choose a natural number $J > 1/\varepsilon$. The sequence $(\boldsymbol{x}_k)_{k>J}$ is still dense in $B$, so by the compactness of $\overline{B}$ there is a finite number of them, $\boldsymbol{x}_{k_i}$ for $i = 1, 2, \ldots, m$, so that $\overline{B} \subset B(\boldsymbol{x}_{k_1}, \varepsilon) \cup \ldots \cup B(\boldsymbol{x}_{k_m}, \varepsilon)$. Let $N = \max(k_1, \ldots, k_m)$. If $k \ge N$ and $\boldsymbol{x} \in B$, then $|\boldsymbol{x} - \boldsymbol{x}_{k_i}| < \varepsilon$ for some $i$, and so

$$\begin{aligned} |u(\boldsymbol{x}) - \tilde{F}_k(\boldsymbol{x})| &\le |u(\boldsymbol{x}) - \tilde{F}_{k_i}(\boldsymbol{x})| \\ &\le |u(\boldsymbol{x}) - \tilde{F}_{k_i}(\boldsymbol{x}_{k_i})| + |\tilde{F}_{k_i}(\boldsymbol{x}_{k_i}) - \tilde{F}_{k_i}(\boldsymbol{x})| \\ &< 1/k_i + L|\boldsymbol{x}_{k_i} - \boldsymbol{x}| \\ &< \varepsilon + L\varepsilon. \end{aligned}$$

Since $L$ was a fixed constant, the uniform convergence is thereby proved. ∎

Unfortunately, the function $u$ produced by this theorem does not necessarily obey the given boundary conditions. For example, let $\Omega = \mathbb{D} \setminus \{\,\mathbf{0}\,\} \subset \mathbb{R}^2$ (here $\mathbb{D} = B(\mathbf{0}, 1)$), and let the boundary value $g$ be given as 0 on $\partial\mathbb{D}$ and 1 on the remaining boundary point $\mathbf{0}$. Clearly, the constant 0 is a subsolution, and (not quite as clear, but true) $1 \wedge (-\varepsilon \ln|x|)$ is a supersolution (see below) for $\varepsilon > 0$. Since no subsolution can be greater, for any $\varepsilon > 0$, every subsolution is $\le 0$. It follows that $u = 0$. It satisfies the boundary condition only on $\partial\mathbb{D}$, but not at the isolated boundary point.

A *supersolution* is a function continuous on $\overline{\Omega}$ and superharmonic in $\Omega$ which is $\ge g$ on $\partial\Omega$. If $f_1$ is a subsolution and $f_2$ is a supersolution, the maximum principle applied to $f_1 - f_2$ (which is subharmonic) shows that $f_1 \le f_2$.

We need a sufficient supply of super- and subsolutions to force $u$ to have the right boundary values.

**Definition.** A *barrier* at $\boldsymbol{y} \in \partial\Omega$ is a function $h \in C(\overline{\Omega})$ which is superharmonic in $\Omega$ with $h(\boldsymbol{y}) = 0$ and $h > 0$ on $\partial\Omega \setminus \{\, \boldsymbol{y} \,\}$. The boundary point $\boldsymbol{y}$ is called *regular* if there exists a barrier at $\boldsymbol{y}$.

**Lemma 23.** *In the setting of Theorem 22, assume that every boundary point of $\Omega$ is regular. Then $\lim_{\boldsymbol{x}\to\boldsymbol{y}} u(\boldsymbol{x}) = g(\boldsymbol{y})$.*

*Proof.* Let $\varepsilon > 0$. Pick $\delta > 0$ so that $\boldsymbol{x} \in \partial\Omega$ and $|\boldsymbol{x} - \boldsymbol{y}| < \delta$ imply $|g(\boldsymbol{x}) - g(\boldsymbol{y})| < \varepsilon$. Let $m > 0$ be the minimum value of $h$ on the set $\{\, \boldsymbol{x} \in \partial\Omega \mid |\boldsymbol{x} - \boldsymbol{y}| \geq \delta \,\}$, and let $M$ be the maximum value of $|g(\boldsymbol{x}) - g(\boldsymbol{y})|$ on the same set. Define functions $f_+$ and $f_-$ by

$$f_\pm(\boldsymbol{x}) = g(\boldsymbol{y}) \pm \varepsilon \pm \frac{M}{m} h(\boldsymbol{x}).$$

Then $f_+$ is a supersolution and $f_-$ is a subsolution, and so $f_- \leq u \leq f_+$. Since $\lim_{\boldsymbol{x}\to\boldsymbol{y}} f_\pm(\boldsymbol{x}) = g(\boldsymbol{y}) \pm \varepsilon$, there is some $\delta' > 0$ so that $\boldsymbol{x} \in \Omega$ and $|\boldsymbol{x} - \boldsymbol{y}| < \delta'$ imply $|u(\boldsymbol{x}) - g(\boldsymbol{y})| < 2\varepsilon$. ∎

From the above lemma, we immediately get the first part of the following theorem:

**Theorem 24.** *Assume $\Omega \subset \mathbb{R}^n$ is a bounded region and that there exists a barrier at every boundar point of $\Omega$. Then the Dirichlet problem has a unique solution for every continuous boundary value.*

*In particular, assume that for every $\boldsymbol{y} \in \partial\Omega$ there is a ball $B \subset \mathbb{R}^n$ with $\overline{B} \cap \overline{\Omega} = \{\, \boldsymbol{y} \,\}$. Then every boundary point of $\Omega$ is regular, and so the Dirichlet problem is solvable on $\Omega$.*

*Proof.* It only remains to prove the last part. If $\boldsymbol{y} \in \partial\Omega$ and the ball $B = B(\boldsymbol{z}, r)$ satisfies the stated property, then

$$h(\boldsymbol{x}) = \Phi(\boldsymbol{y} - \boldsymbol{z}) - \Phi(\boldsymbol{x} - \boldsymbol{z})$$

is a barrier. ∎





**Notation used in this document.** The partial derivative of a function $f$ with respect to $s$ is more often than not shortened from the conventional $\partial f/\partial s$ to $\partial_s f$. The outer unit normal vector of a region $\Omega$ with $C^1$ boundary is written $\boldsymbol{n}$.[4] The *normal derivative* of a $C^1$ function $u$ on $\overline{\Omega}$ is written $\partial_n u = \boldsymbol{n} \cdot u$. The Laplacian of $u$ is written $\Delta u = \nabla \cdot \nabla u$. We use the "barred" integral sign $⨍$ to indicate an *average*, defined to be the ordinary integral divided by the total measure (length, area, or volume) of the region of integration. Thus no matter what we integrate over, $⨍_D 1 \, dx = 1$. We write $A_n$ for the area of the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. Thus a sphere of radius $r$ has area $A_n r^{n-1}$, and the volume of a ball of radius $r$ in $\mathbb{R}^n$ (obtained by integrating $A_n r^{n-1}$) is $A_n r^n/n$. The *Iverson bracket* $[S]$ has the value 1 if the statement $S$ inside is true, and 0 otherwise.

[4] The Greek letter $\boldsymbol{\nu}$ is commonly used for this, but in the font used in this note, that is almost indistuingishable from the latin letter $\boldsymbol{v}$.