Boundary traces

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Functions in the Sobolev space $H^1(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a region, are really only *equivalence classes* of functions, where equivalent functions are equal almost everywhere, so it is not at all clear that it makes sense to speak of *boundary values* of such functions (since the boundary of Ω usually has measure zero). Nevertheless, it turns out it does make sense, though we prefer to use the phrase *boundary trace* to avoid confusion with more conventional boundary values.

We will build on the following result, which we shall not prove here.

Theorem 1 (Meyers–Serrin [1]). *$C^\infty(\mathbb{R}^n) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

By $C^\infty(\mathbb{R}^n) \cap W^{m,p}(\Omega)$, we really mean the space of functions in $W^{m,p}(\Omega)$ which can be extended to C^∞ functions on $\mathbb{R}^n$.

The formulation in [1] is slightly different. The closure in $W^{m,p}(\Omega)$ of $C^\infty(\mathbb{R}^n) \cap W^{m,p}(\Omega)$ was commonly denoted $H^{m,p}(\Omega)$, but Meyers and Serrin proved that these spaces are the same. Hence the amazingly brief title of the paper. Today, as this result is now well known, we use the letter H for a different purpose: $H^m(\Omega) = W^{m,2}(\Omega)$.

The result of interest is the following:

Theorem 2 (The boundary trace theorem). *Assume that the region $\Omega \subset \mathbb{R}^n$ has a piecewise C^1 boundary. Then, for $1 \le p < \infty$, there exists a unique continuous map $\tau\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ so that $\tau(f|_\Omega) = f|_{\partial\Omega}$ for all $f \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega)$.*

Proof of a special case. We prove the theorem for the "half space" $\Omega = \mathbb{R}^{n-1} \times (0,\infty)$, with the boundary $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$, and only for $p = 2$.

But first, we note that the *uniqueness* of the boundary trace map τ follows from Theorem 1, even in the general case. Moreover, in the general case, all that is needed for the *existence* proof is the continuity of the map $f|_\Omega \mapsto f|_{\partial\Omega}$ on $C^\infty(\mathbb{R}^n)$, i.e., an estimate of the form $\|f|_{\partial\Omega}\|_{L^2(\partial\Omega)} \le C\|f|_\Omega\|_{H^1(\Omega)}$, which we shall write more briefly

$$\|f\|_{L^2(\partial\Omega)} \le C\|f\|_{H^1(\Omega)}.$$

We now set about to prove such an estimate for the half space.

First, assume that $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(\boldsymbol{x}, y) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^{n-1}$ and $y \ge 1$. Then

$$-f(\boldsymbol{x},0) = \int_0^1 1 \cdot \partial_y f(\boldsymbol{x},y)\,dy,$$

so by the Cauchy–Schwarz theorem we find

$$|f(\boldsymbol{x},0)|^2 \le 1 \cdot \int_0^1 |\partial_y f(\boldsymbol{x},y)|^2\,dy$$

(where the factor 1 arises as $\int_0^1 1^2\,dy$). Integrating this over $\mathbb{R}^{n-1}$ with respect to $\boldsymbol{x}$ and taking the square root, we conclude that

$$\|f\|_{L^2(\partial\Omega)} \le \|\partial_y f\|_{L^2(\Omega)}.$$

Now, to remove the extra assumption on f, pick some "smooth cutoff function" $\kappa \in C^\infty(\mathbb{R})$ with $\kappa(y) = 1$ for $y \le 0$ and $\kappa(y) = 0$ for $y \ge 1$. For any $f \in C^\infty(\mathbb{R}^n)$, replace f in the above estimate by $f\kappa$, resulting in

$$\begin{aligned}\|f\|_{L^2(\partial\Omega)} &\le \|\partial_y(f\kappa)\|_{L^2(\Omega)}\\ &\le \|f\kappa'\|_{L^2(\Omega)} + \|\kappa\partial_y f\|_{L^2(\Omega)}\\ &\le C_1\|f\|_{L^2(\Omega)} + C_0\|\partial_y f\|_{L^2(\Omega)}\\ &\le C\|f\|_{H^1(\Omega)}.\end{aligned}$$

In the above calculation, $C_0 = \max_y|\kappa(y)|$, $C_1 = \max_y|\kappa'(y)|$, and $C = (C_1^2 + C_0^2)^{1/2}$. This proves the required estimate, and hence the proof for the half space. ∎

The proof of the general case [2] is beyond the scope of this note. In rough outline, however, one proceeds by locally rectifying the boundary – introducing a coordinate change transforming the boundary into the hyperplane $\mathbb{R}^{n-1} \times \{0\}$, thus reducing the general case to the case just proved. A "partition of unity" argument is needed to complete the proof.[1]

Recall the definition of $H_0^1(\Omega)$ as the closure in $H^1(\Omega)$ of $C_c^\infty(\Omega)$. It is an easy exercise to verify that the boundary trace of any function in $H_0^1(\Omega)$ is zero. The converse is also true, though we shall not prove it here: If the boundary trace of some function is zero, then that function belongs to $H_0^1(\Omega)$. This helps to explain the importance of $H_0^1(\Omega)$ in applications.

Bibliography

[1] Norman G. Meyers and James Serrin: $H = W$. Proceedings of the National Academy of Science **51** (1964), 1055–1056.

[2] Lawrence C. Evans and Ronald F. Gariepy: Measure theory and fine properties of functions. CRC Press (1992).

[1] Partition of unity arguments are commonly used in topology and differential geometry to convert local results to global ones.