The energy method for wave equations

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Energy density and energy flow

In this note we study the wave equation

$$u_{tt} - c^2\,\Delta u = f, \tag{1}$$

where the unknown function u is a function of $n+1$ variables: $u(t,\boldsymbol{x})$ for $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$. In general, the "force term" on the right hand side may have the form $f(t,\boldsymbol{x},u,u_t,\nabla u)$, but in most cases it will depend on t and $\boldsymbol{x}$ only.

In the lectures, I will put $c = 1$. The general case can easily be reduced the case $c = 1$ by scaling, but in this note, we retain the general wave speed $c > 0$ anyhow.

If the solution u represents the lateral displacement of a physical membrane, then the quantity $\frac{1}{2}u_t^2$ represents *kinetic energy density*. We would like to understand how energy flows along the membrane, and is exchanged between kinetic and potential energies, however the latter is to be defined. A guiding principle will be that energy flow should be a *vector field*, whose divergence will then correspond to a net energy loss, presumably caused by the external force f.

We start by computing the time derivative of the kinetic energy density, using (1):

$$\partial_t\bigl(\tfrac{1}{2}u_t^2\bigr) = u_t u_t t = c^2 u_t\,\Delta u + u_t f.$$

Here, $u_t f$ on the right hand side might represent the power of the external force f. Can we find a divergence of some vector field hiding inside $u_t\,\Delta u$? A candidate seems to be $u_t\nabla u$, whose divergence is $\nabla\cdot(u_t\nabla u) = u_t\,\Delta u + \nabla u_t\cdot\nabla u$. The final term here is quickly identified as another time derivative, namely of $\frac{1}{2}|\nabla u|^2$ – an excellent candidate for *potential energy*. Putting this together, we have

$$u_t\,\Delta u = \nabla\cdot(u_t\nabla u) - \partial_t\bigl(\tfrac{1}{2}|\nabla u|^2\bigr),$$

which substituted above gives

$$\partial_t\bigl(\tfrac{1}{2}u_t^2 + c^2\tfrac{1}{2}|\nabla u|^2\bigr) = \nabla\cdot(c^2u_t\nabla u) + u_t f.$$

Accordingly, we define the *energy density*

$$e(t,\boldsymbol{x}) = \tfrac{1}{2}\bigl(u_t^2 + c^2|\nabla u|^2\bigr), \tag{2}$$

and the *energy flux density*

$$\boldsymbol{q} = -c^2u_t\nabla u.$$

and conclude that we have the *energy balance equation*

$$e_t + \nabla \cdot \boldsymbol{q} = u_t f. \tag{3}$$

It shows that $\boldsymbol{q}$ can be thought of as a flux corresponding to the energy density e. The source term $u_t f$ represents the power density (work per unit time and volume/area/length, depending on the dimension) of the external force f.

A very useful inequality arises from the trivial inequality $(a-b)^2 \geq 0$ by expanding the square and reorganising it in the form $2ab \leq (a^2+b^2)$. Replacing a and b by their absolute values, we can also write this as $2|ab| \leq (a^2+b^2)$.

In particular, since $|\boldsymbol{q}| = c^2|u_t||\nabla u|$, putting $a = u_t$ and $b = c|\nabla u|$ and taking a quick look at (2) reveals that

$$|\boldsymbol{q}| \leq ce, \tag{4}$$

which we can loosely interpret as follows: *Energy does not travel faster than the wave speed c*. We now make this precise.

Domain of dependence

Consider the PDE (1) with initial data

$$u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad u_t(0,\boldsymbol{x}) = h(\boldsymbol{x}). \tag{5}$$

We consider some $(t_0, \boldsymbol{x}_0)$ with $t_0 > 0$, and ask the following question: What part of the initial data, and force term f (if there is one), influences the solution at $(t_0, \boldsymbol{x}_0)$?

To answer this question, consider the region

$$\Omega_{t_0,\boldsymbol{x}_0} = \big\{(t,\boldsymbol{x}) \,\big|\, 0 < t < t_0,\ |\boldsymbol{x}-\boldsymbol{x}_0| < c(t_0-t)\big\}.$$

Think of it as a ball centered at $\boldsymbol{x}_0$, with a radius shrinking at wave speed c, and which vanishes at $t = t_0$:

$$\Omega_{t_0,\boldsymbol{x}_0} = \bigcup_{0<t<t_0} \{t\} \times B\big(\boldsymbol{x}_0, c(t_0-t)\big).$$

Since the ball shrinks so rapidly, (4) tells us that no new energy can enter from outside, so the total energy should decrease, unless of course the force term adds extra energy.

So we define the energy inside the ball as

$$\mathcal{E}(t) = \int_{B(\boldsymbol{x}_0, c(t_0-t))} e(t,\boldsymbol{x})\, d^n\boldsymbol{x},$$

and compute:

$$\begin{aligned}\frac{d\mathscr{E}}{dt} &= \int_{B(\boldsymbol{x}_0,c(t_0-t))} e_t(t,\boldsymbol{x})\,d^n\boldsymbol{x} - c\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} e(t,\boldsymbol{x})\,dS(\boldsymbol{x})\\ &= \int_{B(\boldsymbol{x}_0,c(t_0-t))} (-\nabla\boldsymbol{q} + u_t f)\,d^n\boldsymbol{x} - c\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} e(t,\boldsymbol{x})\,dS(\boldsymbol{x})\\ &= -\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} (\nu\cdot\boldsymbol{q} + ce)\,d^n\boldsymbol{x} + \int_{B(\boldsymbol{x}_0,c(t_0-t))} u_t f\,d^n\boldsymbol{x}.\end{aligned}$$

Now (4) tells us that the integrand of the boundary integral is nonnegative, so we conclude that

$$\frac{d\mathscr{E}}{dt} \le \int_{B(\boldsymbol{x}_0,c(t_0-t))} u_t f\,d^n\boldsymbol{x}. \tag{6}$$

In the simplest case, $f = 0$ on $\Omega_{t_0,\boldsymbol{x}_0}$, and then (6) tells us that $\mathscr{E}$ is non-increasing.

Theorem 1 *Assume that u_1 and u_2 are classical solutions of*

$$\left.\begin{aligned}\partial_{tt}u_i - c^2\,\Delta u_i &= f_i(t,\boldsymbol{x}),\\ u_i(0,\boldsymbol{x}) &= g_i(\boldsymbol{x}),\\ \partial_t u_i(0,\boldsymbol{x}) &= h_i(\boldsymbol{x}),\end{aligned}\right\}\qquad \text{for } i = 1,2;$$

further assume that $g_1(\boldsymbol{x}) = g_2(\boldsymbol{x})$ and $h_1(\boldsymbol{x}) = h_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in B(\boldsymbol{x}, ct_0)$, and that $f_1(t,\boldsymbol{x}) = f_2(t,\boldsymbol{x})$ for all $(t,\boldsymbol{x}) \in \Omega_{t_0,\boldsymbol{x}_0}$. Then $u_1(t_0,\boldsymbol{x}_0) = u_2(t_0,\boldsymbol{x}_0)$.

Proof: Let $u = u_1 - u_2$. Then u satisfies (1) with $f = f_1 - f_2$ and initial values (5) where $g = g_1 - g_2$ and $h = h_1 - h_2$.

Now apply the calculation on the previous page: Since $g = 0$ and $h = 0$ on $B(\boldsymbol{x}_0)$ and $f = 0$ on $\Omega_{t_0,\boldsymbol{x}_0}$, we find first that $\mathscr{E}(0) = 0$, and second, using (6), that $\mathscr{E}$ is non-increasing. Thus $\mathscr{E}(t) = 0$ for all $t \in [0,t_0]$. In particular $u_t = 0$ inside $\Omega_{t_0,\boldsymbol{x}_0}$, so $u(t_0,\boldsymbol{x}_0) = u(0,\boldsymbol{x}_0) = g(\boldsymbol{x}_0) = 0$. We conclude that $u_1(t_0,\boldsymbol{x}_0) = u_2(t_0,\boldsymbol{x}_0)$. ∎

Solution dependent forcing term. We can prove an analogue of Theorem 1 for more complicated right hand sides:

Theorem 2 *Assume the same conditions as in Theorem 1, with these changes:*

The PDE is now assumed to have the form

$$\partial_{tt}u_i - c^2\,\Delta u_i = f_i\big(t,\boldsymbol{x},u_t(t,\boldsymbol{x}),\nabla u(t,\boldsymbol{x})\big).$$

The conditions on f_1 and f_2 are replaced by the requirement that there exists a constant L so that

$$|f_1(t,\boldsymbol{x},v_1,\boldsymbol{w}_1)-f_2(t,\boldsymbol{x},v_2,\boldsymbol{w}_2)|\le L(|v_1-v_2|+|\boldsymbol{w}_1-\boldsymbol{w}_2|) \tag{7}$$

for all $(t,\boldsymbol{x})\in\Omega_{t_0,\boldsymbol{x}_0}$ and all $v_1,v_2\in\mathbb{R}$ and $\boldsymbol{w}_1,\boldsymbol{w}_2\in\mathbb{R}^n$.

Then, once more, $u_1(t_0,\boldsymbol{x}_0)=u_2(t_0,\boldsymbol{x}_0)$.

Proof: Again, let $u=u_1-u_2$. Then u satisfies the equation

$$u_tt-c^2\,\Delta u=\underbrace{f_1\big(t,\boldsymbol{x},\partial_t u_1(t,\boldsymbol{x}),\nabla u_1(t,\boldsymbol{x})\big)-f_2\big(t,\boldsymbol{x},\partial_t u_2(t,\boldsymbol{x}),\nabla u_2(t,\boldsymbol{x})\big)}_{f(t,\boldsymbol{x})},$$

and so the associated energy satisfies (6):

$$\frac{d\mathscr{E}}{dt}\le\int_{B(\boldsymbol{x}_0,c(t_0-t))}u_tf\,d^n\boldsymbol{x}.$$

Here we note that

$$|f(t,\boldsymbol{x})|\le L\big(|\partial_t u_1-\partial_t u_2|+|\nabla u_1-\nabla u_2|\big)=L\big(|u_t|+|\nabla u|\big),$$

se we get

$$|u_tf|\le L\big(u_t^2+|u_t\nabla u|\big)\le\tfrac{3}{2}L(u_t^2+|\nabla u|^2)=3Le.$$

Now we integrate, and conclude that

$$\frac{d\mathscr{E}}{dt}\le 3L\mathscr{E},$$

and therefore

$$\frac{d}{dt}(e^{-3Lt}\mathscr{E})\le 0$$

From $\mathscr{E}(0)=0$ we then conclude $\mathscr{E}(t_0)=0$, and the proof is completed in the same way as the proof of Theorem 1. ∎

Let us have a second look at the somewhat mysterious equation (7). First, with $v_1=v_2$ and $\boldsymbol{w}_1=\boldsymbol{w}_2$, it implies that

$$f_1(t,\boldsymbol{x},\cdot,\cdot)=f_2(t,\boldsymbol{x},\cdot,\cdot).$$

Given this relation, (7) boils down to *Lipschitz continuity* in the $v=u_t$ and $\boldsymbol{w}=\nabla u$ variables:

$$|f_1(t,\boldsymbol{x},v_1,\boldsymbol{w}_1)-f_1(t,\boldsymbol{x},v_2,\boldsymbol{w}_2)|\le L(|v_1-v_2|+|\boldsymbol{w}_1-\boldsymbol{w}_2|)$$

This holds if the partial derivatives of f_1 with respect to v and $\boldsymbol{w}$ are bounded.

Final remark. Note that we have *not* proved *existence* of solutions to the problem being studied. But we now know that a solution, if it exists, is unique.