First order quasilinear equations

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More general quasilinear equations

Here we look at the general quasilinear equation in two dimensions:

$$au_x + bu_y = c, \tag{1}$$

in which a, b, and c are *really* $a\big(x, y, u(x, y)\big)$, $b\big(x, y, u(x, y)\big)$, and $c\big(x, y, u(x, y)\big)$. (It is the dependence of the coefficients on u that makes the equations *quasi*linear.)

Consider any smooth curve $\big(x(t), y(t)\big)$. Then, assuming u is a classical solution of (1), we put $z(t) = u\big(x(t), y(t)\big)$, and find

$$z'(t) = x'(t)u_x + y'(t)u_y$$

so that if $x'(t) = a$ and $y'(t) = b$, then

$$z' = au_x + bu_y = c.$$

This all means that (x, y, z) satify the *characteristic equations*

$$x'(t) = a(x, y, z), \qquad y'(t) = b(x, y, z), \qquad z'(t) = c(x, y, z). \tag{2}$$

To summarize so far: *Assume that u is a classical solution of* (1). *Through each point in the graph of u,*

$$\text{graph}(u) = \{\, (x, y, z) \mid z = u(x, y) \,\},$$

there passes a characteristic curve $\big(x(t), y(t), z(t)\big)$ solving (2). *Moreover, each such characteristic curve will lie within the graph of u.*

The solution strategy for (1) can now be explained: Since the graph of a classical solution is a union of characteristic curves, we try to construct solutions by putting together characteristic curves.

The graph of u will be two-dimensional, and a characteristic curve is one-dimensional; so it makes sense to use another variable to keep track of the characteristic curves.

In other words, to construct a solution, we look for three functions $x(s, t)$, $y(s, t)$, $z(s, t)$ which parametrize a characteristic curve as function of t for each s. In other words, they should satisfy

$$x_t = a, \qquad y_t = b, \qquad z_t = c$$

where, as always, a, b, and c are considered functions of x, y, and z.[1] Additionally, we assume that these are C^1 functions, and that $\big(x(s,t), y(s,t)\big)$ has a C^1 inverse, mapping (x,y) to (s,t). Then we can define u by

$$u\big(x(s,t), y(s,t)\big) = z(s,t).$$

Differentiating this equation with respect to t yields $x_t u_x + y_t u_y = z_t$, which is the same as (1).

To make this construction more concrete, putting $t = 0$ yields a parametric curve γ in the (x,y)-plane: $\big(x(s,0), y(s,0)\big)$, or, if we add a coordinate, a curve Γ in graph(u) parametrized as $\big(x(s,0), y(s,0), z(s,0)\big)$. The requirement that $\big(x(s,t), y(s,t)\big)$ has a C^1 inverse implies that the matrix $\left(\begin{smallmatrix} x_s & x_t \\ y_s & y_t \end{smallmatrix}\right)$ is non-singular. At $t = 0$ this simply means that Γ is not tangent to the characteristic curves. We refer to this as the *non-characteristic condition.*

We now see what is a natural condition to impose in order to obtain a unique solution to (1): Namely, given a curve γ in the (x,y)-plane, and a function g on γ, assume that the curve Γ given by points (ξ, η, ζ) with (ξ, η) on γ and $\zeta = g(\xi, \eta)$ satisfies the non-characteristic condition. Then (1) has a solution u satisfying the condition

$$u = g \text{ on } \gamma. \tag{3}$$

This solution will exist on a neighbourhood of γ, and be unique there. (Though we must be careful with any end points of γ: They should not be points on γ themselves, as we would then be obliged to extend the solution beyond the end of γ.)

A problem of the form (1) with (3) is called a *Cauchy problem* for (1). More generally, a Cauchy problem for a PDE is the problem of solving the PDE along with certain conditions along a curve, or more generally a hypersurface. Often, as is the case here, the PDE more or less dictates the proper form of the Cauchy problem after some analysis.

[1] Well, they were functions of x, y, and u, right? But in this solution, we should think of u and z as being the same. We write u when we emphasize the solution $u(t,x)$, but z when we think of the characteristic curves, as in $z(t)$ or $z(s,t)$. After you have gained some experience, you may find it easier to forget about z and just write u. But this may be too confusing in the beginning.

Time to collision

Here we consider an IVP (initial value problem) for a quasilinear equation:

$$u_t + a(u)u_x = 0, \qquad u(0,x) = g(x) \tag{4}$$

where the PDE is supposed to hold for $t > 0$, and $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is given. For simplicity, we will assume that a and g are C^1 functions.

The characteristic equations will be

$$x'(t) = a\big(u(t)\big), \qquad u'(t) = 0.$$

To fit with the framework of the more general quasilinear equation, we really should choose a different parameter, say, $\hat{t}$, for the characteristic, and write $t'(\hat{t}) = 1$, $x'(\hat{t}) = a\big(x(\hat{t})\big)$, and $u'(\hat{t}) = 0$. But the first equation, along with the natural choice $t(0) = 0$, yields $t(\hat{t}) = \hat{t}$: So we can (and do) just skip all that, and go for t as the parameter on the characteristic curves right away.

By the second equation, u is constant along any characteristic, and hence so is x', by the first equation. Thus $x(t)$ has the form $x(t) = ct + \xi$ for constants c and ξ. Setting $t = 0$ and recalling that $u(t)$ should really be $u\big(t, x(t)\big)$, we obtain $c = x'(t) = x'(0) = a\big(u(0, x(0))\big) = a\big(g(\xi)\big)$. Writing

$$c(\xi) = a\big(g(\xi)\big),$$

we conclude that the characteristics have the form

$$x = c(\xi)t + \xi, \tag{5}$$

and since u is constant along this characteristic, we must have

$$u(t,x) = g(\xi). \tag{6}$$

To find $u(t,x)$ from (6), we need to solve (5) with respect to ξ for given (t,x). Taking the derivative in (5), we get

$$\frac{\partial x}{\partial \xi} = 1 + tc'(\xi).$$

If $c'(\xi) \geq 0$ for all ξ, it is clear that (5) can be solved with respect to ξ for all $x \in \mathbb{R}$ and $t \geq 0$.

If $c'(\xi) < 0$ for some ξ, however, then we cannot do this when t is too large. Clearly, the critical time in this case is

$$\tau = \frac{-1}{\inf\limits_{\xi \in \mathbb{R}} c'(\xi)}.$$

When $0 < t < \tau$, we can solve (5) for ξ, while when $t > \tau$, we cannot.

Thus τ is the first time of collision of the characteriestics, after which there is no longer a classical solution.

The general quasilinear equation in higher dimensions

We can do the exact same procedure, with the same arguments, in n dimensions as we did in first section of this note. Here we summarize the construction *very* briefly.

A general quasilinear equation then takes the form

$$\boldsymbol{a}\big(\boldsymbol{x}, u(\boldsymbol{x})\big) \cdot \nabla u(\boldsymbol{x}) = c\big(\boldsymbol{x}, u(\boldsymbol{x})\big), \qquad \boldsymbol{x} \in \mathbb{R}^n, \tag{7}$$

with given functions $\boldsymbol{a}$ and c.

The *characteristic equations* become

$$\boldsymbol{x}'(t) = \boldsymbol{a}\big(\boldsymbol{x}(t), z(t)\big), \qquad z'(t) = c\big(\boldsymbol{x}(t), z(t)\big),$$

or written in a more compact form:

$$\boldsymbol{x}' = \boldsymbol{a}(\boldsymbol{x}, z), \qquad z' = c(\boldsymbol{x}, z). \tag{8}$$

Assume now that we are given the PDE (7) with the extra condition

$$u(\boldsymbol{\xi}) = g(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \gamma, \tag{9}$$

where $\gamma \subset \mathbb{R}^n$ is a *hypersurface*, i.e., a surface of dimension $n - 1$. The non-characteristic condition now says that $\boldsymbol{a}\big(\boldsymbol{\xi}, g(\boldsymbol{\xi})\big)$ is not tangent to γ for any $\xi \in \gamma$.

For any $\boldsymbol{\xi} \in \gamma$, standard ODE theory guarantees the existence of a solution of (8) satisfying $\boldsymbol{x}(0) = \boldsymbol{\xi}$ and $z(0) = g(\boldsymbol{\xi})$. Write $\big(\boldsymbol{x}(t;\boldsymbol{\xi}), z(t;\boldsymbol{\xi})\big)$ for this solution, and define

$$u(\boldsymbol{x}\big(t;\boldsymbol{\xi})\big) = z(\boldsymbol{x}\big(t;\boldsymbol{\xi})\big), \qquad \xi \in \gamma,$$

where again, we can show that this is well defined (for t sufficiently close to 0) by using the inverse function theorem. Further, the $n + 1$ variables t, $\boldsymbol{\xi}$ are essentially only n variables, because γ is $(n - 1)$-dimensional. The proof that this produces a classical solution is similar to the 2-dimensional case.