

First order quasilinear equations

Harald Hanche-Olsen

More general quasilinear equations

Here we look at the general quasilinear equation in two dimensions:

$$au_x + bu_y = c, \quad (1)$$

in which a , b , and c are *really* $a(x, y, u(x, y))$, $b(x, y, u(x, y))$, and $c(x, y, u(x, y))$. (It is the dependence of the coefficients on u that makes the equations *quasilinear*.)

Consider any smooth curve $(x(t), y(t))$. Then, assuming u is a classical solution of (1), we put $z(t) = u(x(t), y(t))$, and find

$$z'(t) = x'(t)u_x + y'(t)u_y$$

so that if $x'(t) = a$ and $y'(t) = b$, then

$$z' = au_x + bu_y = c.$$

This all means that (x, y, z) satisfy the *characteristic equations*

$$x'(t) = a(x, y, z), \quad y'(t) = b(x, y, z), \quad z'(t) = c(x, y, z). \quad (2)$$

To summarize so far: *Assume that u is a classical solution of (1). Through each point in the graph of u ,*

$$\text{graph}(u) = \{(x, y, z) \mid z = u(x, y)\},$$

there passes a characteristic curve $(x(t), y(t), z(t))$ solving (2). Moreover, each such characteristic curve will lie within the graph of u .

The solution strategy for (1) can now be explained: Since the graph of a classical solution is a union of characteristic curves, we try to construct solutions by putting together characteristic curves.

The graph of u will be two-dimensional, and a characteristic curve is one-dimensional; so it makes sense to use another variable to keep track of the characteristic curves.

In other words, to construct a solution, we look for three functions $x(s, t)$, $y(s, t)$, $z(s, t)$ which parametrize a characteristic curve as function of t for each s . In other words, they should satisfy

$$x_t = a, \quad y_t = b, \quad z_t = c$$

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where, as always, a , b , and c are considered functions of x , y , and z .¹ Additionally, we assume that these are C^1 functions, and that $(x(s, t), y(s, t))$ has a C^1 inverse, mapping (x, y) to (s, t) . Then we can define u by

$$u(x(s, t), y(s, t)) = z(s, t).$$

Differentiating this equation with respect to t yields $x_t u_x + y_t u_y = z_t$, which is the same as (1).

To make this construction more concrete, putting $t = 0$ yields a parametric curve γ in the (x, y) -plane: $(x(s, 0), y(s, 0))$, or, if we add a coordinate, a curve Γ in $\text{graph}(u)$ parametrized as $(x(s, 0), y(s, 0), z(s, 0))$. The requirement that $(x(s, t), y(s, t))$ has a C^1 inverse implies that the matrix $\begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix}$ is non-singular. At $t = 0$ this simply means that Γ is not tangent to the characteristic curves. We refer to this as the *non-characteristic condition*.

We now see what is a natural condition to impose in order to obtain a unique solution to (1): Namely, given a curve γ in the (x, y) -plane, and a function g on γ , assume that the curve Γ given by points (ξ, η, ζ) with (ξ, η) on γ and $\zeta = g(\xi, \eta)$ satisfies the non-characteristic condition. Then (1) has a solution u satisfying the condition

$$u = g \text{ on } \gamma. \quad (3)$$

This solution will exist on a neighbourhood of γ , and be unique there. (Though we must be careful with any end points of γ : They should not be points on γ themselves, as we would then be obliged to extend the solution beyond the end of γ .)

A problem of the form (1) with (3) is called a *Cauchy problem* for (1). More generally, a Cauchy problem for a PDE is the problem of solving the PDE along with certain conditions along a curve, or more generally a hypersurface. Often, as is the case here, the PDE more or less dictates the proper form of the Cauchy problem after some analysis.

¹Well, they were functions of x , y , and u , right? But in this solution, we should think of u and z as being the same. We write u when we emphasize the solution $u(t, x)$, but z when we think of the characteristic curves, as in $z(t)$ or $z(s, t)$. After you have gained some experience, you may find it easier to forget about z and just write u . But this may be too confusing in the beginning.

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Time to collision

Here we consider an IVP (initial value problem) for a quasilinear equation:

$$u_t + a(u)u_x = 0, \quad u(0, x) = g(x) \quad (4)$$

where the PDE is supposed to hold for $t > 0$, and $g: \mathbb{R} \rightarrow \mathbb{R}$ is given. For simplicity, we will assume that a and g are C^1 functions.

The characteristic equations will be

$$x'(t) = a(u(t)), \quad u'(t) = 0.$$

To fit with the framework of the more general quasilinear equation, we really should choose a different parameter, say, \hat{t} , for the characteristic, and write $t'(\hat{t}) = 1$, $x'(\hat{t}) = a(x(\hat{t}))$, and $u'(\hat{t}) = 0$. But the first equation, along with the natural choice $t(0) = 0$, yields $t(\hat{t}) = \hat{t}$: So we can (and do) just skip all that, and go for t as the parameter on the characteristic curves right away.

By the second equation, u is constant along any characteristic, and hence so is x' , by the first equation. Thus $x(t)$ has the form $x(t) = ct + \xi$ for constants c and ξ . Setting $t = 0$ and recalling that $u(t)$ should really be $u(t, x(t))$, we obtain $c = x'(t) = x'(0) = a(u(0, x(0))) = a(g(\xi))$. Writing

$$c(\xi) = a(g(\xi)),$$

we conclude that the characteristics have the form

$$x = c(\xi)t + \xi, \quad (5)$$

and since u is constant along this characteristic, we must have

$$u(t, x) = g(\xi). \quad (6)$$

To find $u(t, x)$ from (6), we need to solve (5) with respect to ξ for given (t, x) . Taking the derivative in (5), we get

$$\frac{\partial x}{\partial \xi} = 1 + tc'(\xi).$$

If $c'(\xi) \geq 0$ for all ξ , it is clear that (5) can be solved with respect to ξ for all $x \in \mathbb{R}$ and $t \geq 0$.

If $c'(\xi) < 0$ for some ξ , however, then we cannot do this when t is too large. Clearly, the critical time in this case is

$$\tau = \frac{-1}{\inf_{\xi \in \mathbb{R}} c'(\xi)}.$$

When $0 < t < \tau$, we can solve (5) for ξ , while when $t > \tau$, we cannot.

Thus τ is the first time of collision of the characteristics, after which there is no longer a classical solution.

The general quasilinear equation in higher dimensions

We can do the exact same procedure, with the same arguments, in n dimensions as we did in first section of this note. Here we summarize the construction *very* briefly.

A general quasilinear equation then takes the form

$$a(\mathbf{x}, u(\mathbf{x})) \cdot \nabla u(\mathbf{x}) = c(\mathbf{x}, u(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n, \quad (7)$$

with given functions a and c .

The *characteristic equations* become

$$\mathbf{x}'(t) = a(\mathbf{x}(t), z(t)), \quad z'(t) = c(\mathbf{x}(t), z(t)),$$

or written in a more compact form:

$$\mathbf{x}' = a(\mathbf{x}, z), \quad z' = c(\mathbf{x}, z). \quad (8)$$

Assume now that we are given the PDE (7) with the extra condition

$$u(\xi) = g(\xi), \quad \xi \in \gamma, \quad (9)$$

where $\gamma \subset \mathbb{R}^n$ is a *hypersurface*, i.e., a surface of dimension $n - 1$. The non-characteristic condition now says that $a(\xi, g(\xi))$ is not tangent to γ for any $\xi \in \gamma$.

For any $\xi \in \gamma$, standard ODE theory guarantees the existence of a solution of (8) satisfying $\mathbf{x}(0) = \xi$ and $z(0) = g(\xi)$. Write $(\mathbf{x}(t; \xi), z(t; \xi))$ for this solution, and define

$$u(\mathbf{x}(t; \xi)) = z(\mathbf{x}(t; \xi)), \quad \xi \in \gamma,$$

where again, we can show that this is well defined (for t sufficiently close to 0) by using the inverse function theorem. Further, the $n + 1$ variables t, ξ are essentially only n variables, because γ is $(n - 1)$ -dimensional. The proof that this produces a classical solution is similar to the 2-dimensional case.