The energy method for wave equations

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Energy density and energy flow

In this note we study the wave equation

$$u_{tt} - c^2 \triangle u = f, \tag{1}$$

where the unknown function u is a function of $n+1$ variables: $u(t,\boldsymbol{x})$ for $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$. In general, the "force term" on the right hand side may have the form $f(t,\boldsymbol{x},u,u_t,\nabla u)$, but in most cases it will depend on t and $\boldsymbol{x}$ only.

In the lectures, I will put $c = 1$. The general case can easily be reduced the case $c = 1$ by scaling, but in this note, we retain the general wave speed $c > 0$ anyhow.

Assuming u satisfies (1), we associate with u an *energy density*

$$e(t,\boldsymbol{x}) = \tfrac{1}{2}\big(u_t^2 + c^2|\nabla u|^2\big), \tag{2}$$

from which we compute (using (1))

$$e_t = u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t = c^2 u_t \triangle u + u_t f + c^2 \nabla u \cdot \nabla u_t,$$

which we can write in the form

$$e_t + \nabla \cdot \boldsymbol{q} = u_t f, \tag{3}$$

where the *energy flux density* is

$$\boldsymbol{q} = -c^2 u_t \nabla u.$$

We can think of (3) as an *energy balance equation*, where the right hand side $c^2 u_t f$ represent the work done by the force f. (Note that in physics, force times velocity equals power.)

A very useful inequality arises from the trivial inequality $(a-b)^2 \ge 0$ by expanding the square and reorganising it in the form $2ab \le (a^2+b^2)$. Replacing a and b by their absolute values, we can also write this as $2|ab| \le (a^2+b^2)$.

In particular, since $|\boldsymbol{q}| = c^2|u_t||\nabla u|$, putting $a = u_t$ and $b = c|\nabla u|$ and taking a quick look at (2) reveals that

$$|\boldsymbol{q}| \le ce, \tag{4}$$

which we can loosely interpret as follows: *Energy does not travel faster than the wave speed c*. We now make this precise.

Domain of dependence

Consider the PDE (1) with initial data

$$u(0,\boldsymbol{x}) = g(\boldsymbol{x}), \quad u_t(0,\boldsymbol{x}) = h(\boldsymbol{x}). \tag{5}$$

We consider some $(t_0,\boldsymbol{x}_0)$ with $t_0 > 0$, and ask the following question: What part of the initial data, and force term f (if there is one), influences the solution at $(t_0,\boldsymbol{x}_0)$?

To answer this question, consider the region

$$\Omega_{t_0,\boldsymbol{x}_0} = \big\{(t,\boldsymbol{x}) \,\big|\, 0 < t < t_0,\ |\boldsymbol{x}-\boldsymbol{x}_0| < c(t_0-t)\big\}.$$

Think of it as a ball centered at $\boldsymbol{x}_0$, with a radius shrinking at wave speed c, and which vanishes at $t = t_0$:

$$\Omega_{t_0,\boldsymbol{x}_0} = \bigcup_{0<t<t_0} \{t\} \times B\big(\boldsymbol{x}_0, c(t_0-t)\big).$$

Since the ball shrinks so rapidly, (4) tells us that no new energy can enter from outside, so the total energy should decrease, unless of course the force term adds extra energy.

So we define the energy inside the ball as

$$\mathscr{E}(t) = \int_{B(\boldsymbol{x}_0,c(t_0-t))} e(t,\boldsymbol{x})\,d^n\boldsymbol{x},$$

and compute:

$$\begin{aligned}
\frac{d\mathscr{E}}{dt} &= \int_{B(\boldsymbol{x}_0,c(t_0-t))} e_t(t,\boldsymbol{x})\,d^n\boldsymbol{x} - c\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} e(t,\boldsymbol{x})\,dS(\boldsymbol{x}) \\
&= \int_{B(\boldsymbol{x}_0,c(t_0-t))} (-\nabla \boldsymbol{q} + u_t f)\,d^n\boldsymbol{x} - c\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} e(t,\boldsymbol{x})\,dS(\boldsymbol{x}) \\
&= -\int_{\partial B(\boldsymbol{x}_0,c(t_0-t))} (\nu \cdot \boldsymbol{q} + ce)\,d^n\boldsymbol{x} + \int_{B(\boldsymbol{x}_0,c(t_0-t))} u_t f\,d^n\boldsymbol{x}.
\end{aligned}$$

Now (4) tells us that the integrand of the boundary integral is nonnegative, so we conclude that

$$\frac{d\mathscr{E}}{dt} \le \int_{B(\boldsymbol{x}_0,c(t_0-t))} u_t f\,d^n\boldsymbol{x}. \tag{6}$$

In the simplest case, $f = 0$ on $\Omega_{t_0,\boldsymbol{x}_0}$, and then (6) tells us that $\mathscr{E}$ is non-increasing.

Theorem 1 *Assume that u_1 and u_2 are classical solutions of*

$$\left.\begin{aligned} \partial_{tt}u_i - c^2 \triangle u_i &= f_i(t,\boldsymbol{x}), \\ u_i(0,\boldsymbol{x}) &= g_i(\boldsymbol{x}), \\ \partial_t u_i(0,\boldsymbol{x}) &= h_i(\boldsymbol{x}), \end{aligned}\right\} \quad \text{for } i = 1,2;$$

further assume that $g_1(\boldsymbol{x}) = g_2(\boldsymbol{x})$ and $h_1(\boldsymbol{x}) = h_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in B(\boldsymbol{x}, ct_0)$, and that $f_1(t,\boldsymbol{x}) = f_2(t,\boldsymbol{x})$ for all $(t,\boldsymbol{x}) \in \Omega_{t_0,\boldsymbol{x}_0}$. Then $u_1(t_0,\boldsymbol{x}_0) = u_2(t_0,\boldsymbol{x}_0)$.

Proof: Let $u = u_1 - u_2$. Then u satisfies (1) with $f = f_1 - f_2$ and initial values (5) where $g = g_1 - g_2$ and $h = h_1 - h_2$.

Now apply the calculation on the previous page: Since $g = 0$ and $h = 0$ on $B(\boldsymbol{x}_0)$ and $f = 0$ on $\Omega_{t_0,\boldsymbol{x}_0}$, we find first that $\mathscr{E}(0) = 0$, and second, using (6), that $\mathscr{E}$ is non-increasing. Thus $\mathscr{E}(t) = 0$ for all $t \in [0,t_0]$. In particular $u_t = 0$ inside $\Omega_{t_0,\boldsymbol{x}_0}$, so $u(t_0,\boldsymbol{x}_0) = u(0,\boldsymbol{x}_0) = g(\boldsymbol{x}_0) = 0$. We conclude that $u_1(t_0,\boldsymbol{x}_0) = u_2(t_0,\boldsymbol{x}_0)$. ∎

Solution dependent forcing term. We can prove an analogue of Theorem 1 for more complicated right hand sides:

Theorem 2 *Assume the same conditions as in Theorem 1, with these changes:*

The PDE is now assumed to have the form

$$\partial_{tt}u_i - c^2 \triangle u_i = f_i\big(t,\boldsymbol{x},u_t(t,\boldsymbol{x}),\nabla u(t,\boldsymbol{x})\big).$$

The conditions on f_1 and f_2 are replaced by the requirement that there exists a constant L so that

$$|f_1(t,\boldsymbol{x},v_1,\boldsymbol{w}_1) - f_2(t,\boldsymbol{x},v_2,\boldsymbol{w}_2)| \le L(|v_1 - v_2| + |\boldsymbol{w}_1 - \boldsymbol{w}_2|) \tag{7}$$

for all $(t,\boldsymbol{x}) \in \Omega_{t_0,\boldsymbol{x}_0}$ and all $v_1, v_2 \in \mathbb{R}$ and $\boldsymbol{w}_1, \boldsymbol{w}_1 \in \mathbb{R}^n$.

Then, once more, $u_1(t_0,\boldsymbol{x}_0) = u_2(t_0,\boldsymbol{x}_0)$.

Proof: Again, let $u = u_1 - u_2$. Then u satisfies the equation

$$u_t t - c^2 \triangle u = \underbrace{f_1\big(t,\boldsymbol{x},\partial_t u_1(t,\boldsymbol{x}),\nabla u_1(t,\boldsymbol{x})\big) - f_2\big(t,\boldsymbol{x},\partial_t u_2(t,\boldsymbol{x}),\nabla u_2(t,\boldsymbol{x})\big)}_{f(t,\boldsymbol{x})},$$

and so the associated energy satisfies (6):

$$\frac{d\mathscr{E}}{dt} \le \int_{B(\boldsymbol{x}_0,c(t_0-t))} u_t f \, d^n\boldsymbol{x}.$$

Here we note that

$$|f(t,\boldsymbol{x})| \le L\big(|\partial_t u_1 - \partial_t u_2| + |\nabla u_1 - \nabla u_2|\big) = L\big(|u_t| + |\nabla u|\big),$$

se we get

$$|u_t f| \le L\big(u_t^2 + |u_t \nabla u|\big) \le \tfrac{3}{2}L(u_t^2 + |\nabla u|^2) = 3Le.$$

Now we integrate, and conclude that

$$\frac{d\mathscr{E}}{dt} \le 3L\mathscr{E},$$

and therefore

$$\frac{d}{dt}(e^{-3Lt}\mathscr{E}) \le 0$$

From $\mathscr{E}(0) = 0$ we then conclude $\mathscr{E}(t_0) = 0$, and the proof is completed in the same way as the proof of Theorem 1. ∎

Let us have a second look at the somewhat mysterious equation (7). First, with $v_1 = v_2$ and $\boldsymbol{w}_1 = \boldsymbol{w}_2$, it implies that

$$f_1(t,\boldsymbol{x},\cdot,\cdot) = f_2(t,\boldsymbol{x},\cdot,\cdot).$$

Given this relation, (7) boils down to *Lipschitz continuity* in the $v = u_t$ and $\boldsymbol{w} = \nabla u$ variables:

$$|f_1(t,\boldsymbol{x},v_1,\boldsymbol{w}_1) - f_1(t,\boldsymbol{x},v_2,\boldsymbol{w}_2)| \le L(|v_1 - v_2| + |\boldsymbol{w}_1 - \boldsymbol{w}_2|)$$

This holds if the partial derivatives of f_1 with respect to v and $\boldsymbol{w}$ are bounded.

Final remark. Note that we have *not* proved *existence* of solutions to the problem being studied. But we now know that a solution, if it exists, is unique.