

Week 44

10.5 $u(x) = r^\alpha$ on \mathbb{D}

(a) Classical derivatives: $u_{x_j} = \alpha r^{\alpha-1} \frac{x_j}{r} = \alpha x_j r^{\alpha-2}$

(b) We find $|u_{x_j}| \leq |\alpha| r^{\alpha-1}$, and for $\alpha > -1$ we get

$$\int_{\mathbb{D}} |u_{x_j}| d^2x \leq |\alpha| \int_{\mathbb{D}} r^{\alpha-1} d^2x = 2\pi |\alpha| \int_0^1 r^{\alpha-1} \cdot r dr = 2\pi |\alpha| \int_0^1 r^\alpha dr < \infty$$

For $\psi \in C_c^\infty(\mathbb{D})$ we get

$$\int_{\mathbb{D}} \alpha x_j r^{\alpha-2} \psi(x) d^2x = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{D} \setminus B(0, \varepsilon)} \alpha x_j r^{\alpha-2} \psi(x) d^2x$$

$$= \lim_{\varepsilon \downarrow 0} \left(- \int_{\partial B(0, \varepsilon)} r^\alpha \psi \frac{x_j}{\varepsilon} ds - \int_{\mathbb{D} \setminus B(0, \varepsilon)} r^\alpha \psi_{x_j} d^2x \right)$$

$$\left| r^\alpha \psi \frac{x_j}{\varepsilon} \right| \leq \varepsilon^\alpha \max |\psi|$$

on $\partial B(0, \varepsilon)$, so the first integral has absolute value $\leq 2\pi \varepsilon^{\alpha+1} \max |\psi| \rightarrow 0$

$$\downarrow (\varepsilon \downarrow 0)$$

$$\int_{\mathbb{D}} r^\alpha \psi_{x_j} d^2x$$

(c) $r^\alpha \in L^2 \Leftrightarrow r^{2\alpha} \in L^1 \Leftrightarrow 2\alpha > -2 \Leftrightarrow \alpha > -1$

$$\int_{\mathbb{D}} (u_{x_1}^2 + u_{x_2}^2) d^2x = \int_{\mathbb{D}} \alpha^2 r^{2(\alpha-1)} d^2x < \infty \Leftrightarrow 2(\alpha-1) > -2 \Leftrightarrow \alpha > 0.$$

10.7 When $u \in C^2$, $\int_{\Omega} \nabla u \cdot \nabla \psi d^2x = - \int_{\Omega} \psi \Delta u d^2x + \int_{\partial \Omega} \psi \partial_\nu u dS$

so the assumption gives

$$\int_{\Omega} (-\Delta u - f) \psi d^2x + \int_{\partial \Omega} \psi \partial_\nu u dS = 0$$

First taking $\psi \in C_c^\infty(\Omega)$, we conclude $-\Delta u - f = 0$.

Then we are left with $\int_{\partial \Omega} \psi \partial_\nu u dS = 0$ for all $\psi \in C^0(\bar{\Omega})$,

and this implies $\partial_\nu u = 0$.

For the final exam problem, a solution is already available.