

## Week 42

7.1 Parallelogram law:  $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$

(a) Expend:  $\|v \pm w\|^2 = \|v\|^2 \pm 2 \operatorname{Re}\langle v, w \rangle + \|w\|^2$  and add.

(b)  $f = \chi_{[0,2]} \quad g = \chi_{[0,1]} - \chi_{[1,2]} \quad \|f\|_p = \|g\|_p = 2^{\frac{1}{p}}$

$$f+g = 2\chi_{[0,1]}, \quad \|f+g\|_p = 2 \quad \|f-g\|_p = 2\chi_{[1,2]}, \quad \|f-g\|_p = 2$$

$$\|f+g\|_p^2 + \|f-g\|_p^2 = 8, \quad 2\|f\|_p^2 + 2\|g\|_p^2 = 4 \cdot 2^{\frac{2}{p}} = 2^{\frac{2+2}{p}} \neq 8 \text{ if } p \neq 2.$$

(c) The functions from (b) work here, too:

$$\|f\|_\infty = \|g\|_\infty = 1, \quad \|f+g\|_\infty = \|f-g\|_\infty = 2$$

7.2  $f_n(x) = ne^{-n^2x} \quad [x \geq 0]$  gives

$$\|f_n\|_1 = \int_0^\infty n e^{-n^2x} dx = n/n^2 \rightarrow 0$$

$$\|f_n\|_2^2 = \int_0^\infty n^2 e^{-2n^2x} dx = \frac{n^2}{2n^2} = \frac{1}{2} \rightarrow 0$$

7.3  $\|g_n\|_2^2 = \int_0^n n^{-2} dx = n \cdot n^{-2} = 1/n \rightarrow 0$

$$\|g_n\|_1 = \int_0^n n^{-1} dx = n \cdot n^{-1} = 1 \rightarrow 0$$

7.5 Use Cauchy-Schwarz!

The constant function 1 is in  $L^2$ :  $\|1\|_2 = \left( \int_{\Omega} 1 dx \right)^{1/2} = m_{\Omega}(\Omega)^{1/2}$

$$\text{so } \|f\|_1 = \int_{\Omega} |f| dx = \langle |f|, 1 \rangle \leq \|f\|_2 \cdot \|1\|_2$$

The sequence in 7.3 provides a counterexample in an unbounded domain.

7.6  $u_t - \Delta u = 0$  in  $(0, \infty) \times \Omega$ ,  $\gamma(t) = \int_{\Omega} u^2 d^n x$

$$(a) |\gamma'(t)| = \left| 2 \int_{\Omega} u u_t d^n x \right| \leq 2 \|u\|_2 \|u_t\|_2$$

$$\text{Square it: } \gamma'(t)^2 \leq 4 \gamma(t) \int_{\Omega} |u_t|^2 d^n x$$

$$(b) \gamma'' = 2 \int_{\Omega} (u_t^2 + u u_{tt}) d^n x$$

$$\begin{aligned} \text{and } \int_{\Omega} u u_{tt} d^n x &= \int_{\Omega} u \Delta u_t d^n x = \int_{\Omega} u_t \Delta u d^n x + \int_{\Omega} (\underbrace{u \partial_n u_t - u_t \partial_n u}_{=0}) d^n x \\ &= \int_{\Omega} u_t^2 d^n x \end{aligned}$$

$$\text{so } \gamma'' = 4 \int_{\Omega} u_t^2 d^n x - \text{ and therefore } (\gamma')^2 \leq \gamma \gamma''$$

$$(c) (\ln \gamma)' = \frac{\gamma'}{\gamma} \quad \ln(\gamma)'' = \frac{\gamma \gamma'' - (\gamma')^2}{\gamma^2} \geq 0$$

(d) Do the converse: From (c) we get that, if  $\gamma(t) > 0$  then  $\gamma(t') > 0$  for any  $t' > t$  (think of the heat equation with initial data given at  $t \dots$ ). And so  $\gamma(T) > 0$ .

7.7

$$(a) g(x) = r^\gamma \quad [r \leq 1] : \int_{B^n} g^p d^n x = \int_0^1 \int_{\{x \cdot e_r = r\}} r^{\gamma p} dS dr = A_n \int_0^1 r^{n-1+\gamma p} dr$$

$$< \infty \Leftrightarrow n-1+\gamma p > -1 \Leftrightarrow \gamma p > -n$$

$$(b) h(x) = r^\gamma \quad [r \geq 1] : \int_{B^n} g^p d^n x = A_n \int_1^\infty r^{n-1+\gamma p} dr$$

$$< \infty \Leftrightarrow n-1+\gamma p < -1 \Leftrightarrow \gamma p < -n$$

Polarization identity:

$$\begin{aligned} & \sum_{k=0}^3 i^k \|u + i^k v\|^2 \\ &= \sum_{k=0}^3 i^k \langle u + i^k v, u + i^k v \rangle \\ &= \sum_{k=0}^3 i^k (\langle u, u \rangle + i^k \langle v, u \rangle - i^k \langle u, v \rangle + \langle v, v \rangle) \\ &= \sum_{k=0}^3 (i^k \langle u, u \rangle + i^{2k} \langle v, u \rangle + \langle u, v \rangle + i^k \langle v, v \rangle) \\ &= 4 \langle u, v \rangle \end{aligned}$$

because  $\sum_{k=0}^3 i^k = 0$ ,  $\sum_{k=0}^3 i^{2k} = \sum_{k=0}^3 (-1)^k = 0$ .