

Week 39

From the note on the parabolic maximum principle

Exercise 1

$$\begin{aligned} \text{Put } u = u_1 - u_2: \quad u_t - \Delta u &= f_1 - f_2 =: f \\ u = g_1 - g_2 &=: g \quad \text{on } \partial\Omega \end{aligned}$$

$$\text{Put } \varphi = \sup_{\Omega_T} |f|, \quad \gamma = \sup_{\Gamma} |g|$$

$$\text{Take the hint: Put } v = u - \varphi t$$

$$\text{then } v_t - \Delta v = u_t - \Delta u - \varphi = f - \varphi \leq 0$$

$$\text{so } u = v + \varphi t \leq (\max_{\Gamma} v) + \varphi t$$

$$\leq (\max_{\Gamma} g) + \varphi t \leq \gamma + \varphi t$$

Next, swap u_1 and u_2 . Then u becomes $-u$, so $-u \leq \gamma + \varphi t$.

Combined: $|u| \leq \gamma + \varphi t$. The problem asked for $|u| \leq \gamma + c\sqrt{T}$; but this is better.

Exercise 2 Like for the heat equation, we put $v = u - \varepsilon t$ and prove that v cannot achieve its maximum on $\Omega_T \setminus \Gamma$. In fact, if v does achieve a maximum at $(t, \vec{x}) \in \Omega_T \setminus \Gamma$ then $v_t \geq 0$, $\nabla v = 0$, and $Hv \leq 0$ at the given point.

(By $Hv \leq 0$, I mean that $-Hv$ is positive semidefinite.)

$$\text{Now } v_t - \nabla \cdot (A \nabla v) = -\varepsilon + u_t - \nabla \cdot (A \nabla v) = -\varepsilon < 0,$$

and since $v_t \geq 0$ at the given point, $\nabla \cdot (A \nabla v) > 0$ there.

$$\text{But } \nabla \cdot (A \nabla v) = \langle A, Hv \rangle_F \leq 0 \text{ since } A \geq 0 \text{ and } Hv \leq 0.$$

This is a contradiction.

Exercise 3 Just follow the steps of Ex. 2. At any maximum, we find $\nabla u = 0$, and therefore $b(\nabla u) = 0$,

Extra exercise, after correction:

$$\begin{aligned}\frac{d}{dt} \frac{1}{2} \int_{\Omega} u^2 d^n x &= \int_{\Omega} u u_t d^n x = \int_{\Omega} u (\nabla \cdot (A \nabla u) - \vec{b} \cdot \nabla u) d^n x \\ &= - \int_{\Omega} (\nabla u \cdot A \nabla u + \vec{b} \cdot \nabla u) d^n x + \underbrace{\int_{\partial \Omega} u \vec{n} \cdot A \nabla u dS}_{\text{zero!}} \\ &\leq - \int_{\Omega} (\lambda |\nabla u|^2 + u \vec{b} \cdot \nabla u) d^n x\end{aligned}$$

where $\lambda > 0$ is the smallest eigenvalue of A .

Next, we use the Peter-Paul inequality: $2ab \leq \epsilon a^2 + b^2/\epsilon$

$$|u \vec{b} \cdot \nabla u| \leq |u \vec{b}| \cdot |\nabla u| \leq \lambda |\nabla u|^2 + |\vec{b}|^2 / 4\lambda$$

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} u^2 d^n x \leq \frac{|\vec{b}|^2}{4\lambda} \int_{\Omega} u^2 d^n x, \text{ so } \gamma(x) = \frac{1}{2} \int_{\Omega} u^2 d^n x$$

$$\text{satisfies } \gamma'(t) \leq \frac{|\vec{b}|^2}{2\lambda} \gamma(t)$$

$$\text{and therefore } \frac{d}{dt} \left(\exp \left(-\frac{|\vec{b}|^2}{2\lambda} t \right) \gamma \right) \leq 0.$$

In particular, if $\gamma(0)=0$ then $\gamma(t)=0$ for all $t>0$.