Weak maximum principle for the heat equation

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In this note, we consider the standard heat equation

$$u_t - \triangle u = 0 \qquad \text{in } \Omega_T$$

in which $\Omega_T = (0, T) \times \Omega$ where $\Omega \subset \mathbb{R}^n$ is a *bounded* region, $T > 0$, and

$$u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T).$$

We think of Ω_T as an open *cylinder* with base Ω and height T. Its closure is a closed cylinder: $\overline{\Omega_T} = [0, T] \times \overline{\Omega}$.

Definition. The *parabolic boundary* of Ω_T is the set

$$\Gamma = (\{0\} \times \overline{\Omega}) \cup ([0, t] \times \partial\Omega).$$

Clearly, Γ is contained in the normal boundary $\partial\Omega_T$; the difference is

$$\partial\Omega_T \setminus \Gamma = \{T\} \times \Omega.$$

We call $\{T\} \times \Omega$ the *final boundary* of Ω_T (nonstandard nomenclature).

Observation. If a C^2 function v has a maximum at some point in Ω_T, then $v_t = 0$ and $\triangle v \le 0$ at that point, so we get $v_t - \triangle v \ge 0$ there. Moreover, this holds at the final boundary as well, the only difference being that there, we can only conclude $v_t \ge 0$ and $\triangle v \le 0$. In other words,

$$v_t - \triangle v \ge 0 \quad \text{at any maximum in } \overline{\Omega_T} \setminus \Gamma.$$

We must face a minor technical glitch: The above statement requires that v is C^2 up to and including the final boundary of Ω_T. This complicates the proof of the following theorem, but only a little.

Theorem 1 (The weak maximum principle). *Assume that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies*

$$u_t - \triangle u \le 0.$$

Then $u(t, \boldsymbol{x}) \le \max_\Gamma u$ for all $(t, \boldsymbol{x}) \in \overline{\Omega_T}$. In other words, u achieves its maximum on the parabolic boundary.

Proof. First, to deal with the "minor technical glitch" mentioned above, we shall strengthen the assumptions somewhat, and assume that $u \in C^2\big((0,T] \times \Omega\big)$. We will remove this extra assumption at the end.

Now let $\varepsilon > 0$, and put $v(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) - \varepsilon t$. Then $v_t - \triangle v \le -\varepsilon < 0$, and so it follows *immediately* from the Observation above that v cannot achieve its maximum anywhere outside Γ. On the other hand, since v is continuous and $\overline{\Omega_T}$ is compact, v does have a maximum in $\overline{\Omega_T}$, and so we must conclude that $v(t,\boldsymbol{x}) \le \max_\Gamma v$ for any $(t,\boldsymbol{x}) \in \overline{\Omega_T}$. But then $u(t,\boldsymbol{x}) = v(t,\boldsymbol{x}) + \varepsilon t \le \max_\Gamma v + \varepsilon T \le \max_\Gamma u + \varepsilon T$. Since this holds for any $\varepsilon > 0$, it finally follows that $u(t,\boldsymbol{x}) \le \max_\Gamma u$, and the proof is complete, with the strengthened assumptions.

We now drop the requirement that $u \in C^2\big((0,T] \times \Omega\big)$. However, it is *still* true that $u \in C^2\big((0,T'] \times \Omega\big)$, for any $T' < T$, so the first part shows that $u(t,\boldsymbol{x}) \le \max_{\Gamma_{T'}} u$ for all $(t,\boldsymbol{x}) \in \overline{\Omega_{T'}}$. Here $\Gamma_{T'}$ is tbe parabolic boundary of $\Omega_{T'}$. But $\Gamma_{T'} \subset \Gamma$, so we also have $u(t,\boldsymbol{x}) \le \max_\Gamma u$. For any $t < T$, we can pick T' with $t < T' < T$, so the inequality holds. Finally, it also holds for $t = T$, since u is continuous on $\overline{\Omega_T}$. This, at last, completes the proof. ∎

It should come as no surprise that there is also a *minimum* principle. It is proved by replacing u by $-u$ in Theorem 1.

Corollary 2 (The weak minimum principle)**.** *Assume that $u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big)$ satisfies*

$$u_t - \triangle u \ge 0.$$

Then $u(t,\boldsymbol{x}) \ge \min_\Gamma u$ for all $(t,\boldsymbol{x}) \in \overline{\Omega_T}$. In other words, u achieves its minimum on the parabolic boundary.

We will mostly be concerned with solutions of the heat equation $u_t - \triangle u = 0$, and for these, both the maximum principle and the minimum principle can be used. But we may also wish to study inhomogeneous equations $u_t - \triangle u = f$, and if f has a definite sign, one or the other principle will apply.

Corollary 3 (Uniqueness for the heat equation)**.** *There exists at most one solution $u \in C\big(\overline{\Omega_T}\big) \cap C^2\big(\Omega_T\big)$ to the problem*

$$\begin{aligned} u_t - \triangle u &= f \quad \text{in } \Omega_T, \\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

Here, f and g are given functions on Ω_T and Γ, respectively. (Thus g combines initial values and boundary values in one function.)

Proof. Let u be the difference between two solutions to this problem: Then u solves the same problem, but with $f = 0$ and $g = 0$. Thus u achieves both its minimum and maximum on Γ, but $u = 0$ there, so $u = 0$ everywhere. ∎

The following corollary is proved in essentially the same way, by applying the minimum and maximum principles to $u_1 - u_2$. Note that it immediately implies the preceding corollary by taking $g_1 = g_2$.

Corollary 4 (Continuous dependence on data). *Let u_1 and u_2 satisfy*

$$\left.\begin{aligned} u_{it} - \triangle u_i &= f \quad \textit{in } \Omega_T, \\ u_i &= g_i \quad \textit{on } \Gamma, \end{aligned}\right\} \quad \textit{for } i = 1, 2.$$

Then $|u_1 - u_2| \le \max_\Gamma |g_1 - g_2|$.

Exercise 1 (Continuous dependence on data, improved)**.** Assume that u_1 and u_2 satisfy

$$\left.\begin{aligned} u_{it} - \triangle u_i &= f_i \quad \text{in } \Omega_T, \\ u_i &= g_i \quad \text{on } \Gamma, \end{aligned}\right\} \quad \text{for } i = 1, 2.$$

Let $\varphi = \sup_{\Omega_T} |f_1 - f_2|$ and $\gamma = \max_\Gamma |g_1 - g_2|$, and show that $|u_1 - u_2| \le \gamma + \varphi T$.

Note that for any t, we can pick $T = t$, so we really get $|u_1 - u_2| \le \gamma + \varphi t$.

Hint: Apply the maximum principle to $u_1 - u_2 - \varphi t$ and $u_2 - u_1 - \varphi t$.

Exercise 2. Some definitions:

The *Hessian* of u is the (symmetric!) $n \times n$ matrix $\mathrm{H}u$ with entries $u_{x_i x_j}$.

The *Frobenius inner product* of two real matrices A and B is

$$\langle A, B \rangle_{\mathrm{F}} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} = \operatorname{tr}(A^T B).$$

Show that the maximum (and minimum) principle continues to hold if $u_t - \triangle u$ is replaced by the more general

$$u_t - \langle A, \mathrm{H}u \rangle_{\mathrm{F}}$$

where the real matrix A is symmetric and positive definite.

Here are some ingredients of the proof:

- At an interior maximum point, $-\mathrm{H}u$ is positive semidefinite, i.e., $\boldsymbol{y}^T \mathrm{H} \boldsymbol{y} \le 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$. (Short proof: Take the second derivative of $u(\boldsymbol{x} + s\boldsymbol{y})$ with respect to s where x is a maximum point, and put $s = 0$.)
- It is known that if A and B are positive semidefinite, then $\langle A, B \rangle_{\mathrm{F}} \ge 0$. (Short proof: Since A is symmetric, we can write $\langle A, B \rangle_{\mathrm{F}} = \operatorname{tr}(AB)$. A will have a positive semidefinite square root $A^{1/2}$. A standard result on the trace gives $\operatorname{tr}(AB) = \operatorname{tr}(A^{1/2} A^{1/2} B) = \operatorname{tr}(A^{1/2} B A^{1/2})$, but $A^{1/2} B A^{1/2}$ is positive semidefinite, and such matrices have nonnegative trace.)

Exercise 3. Show that the maximum (and minimum) principle continues to hold if $u_t - \triangle u$ is replaced by the even more general

$$u_t - \langle A, \mathrm{H}u\rangle_{\mathrm{F}} + b(\nabla u),$$

where the real matrix A is symmetric and positive definite, provided b satisfies $b(\mathbf{0}) = 0$. (For a simple and common example, let $b(\nabla u) = \boldsymbol{b} \cdot \nabla u$.)