Harmonicfunctionology

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The Laplace operator on $\mathbb{R}^n$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

plays a rôle in the wave and heat equations, but even more fundamentally, in the *Laplace equation*

$$\Delta u = 0 \tag{1}$$

and the *Poisson equation* $-\Delta u = f$ where f is a given function.

A C^2 solution of (1) is called *harmonic*. (Later, we will find that harmonic functions are in fact infinitely differentiable.)

Much is gained from the trivial observation that $\Delta u = \nabla \cdot \nabla u$ together with various applications of the divergence theorem or two of its corollaries, Green's first and second identities.

Let's get started by simply integrating Δu over a bounded domain ω with piecewise C^1 boundary:

$$\int_\omega \Delta u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_\omega \nabla \cdot \nabla u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\omega} \partial_\nu u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}). \tag{2}$$

This immediately proves

Proposition 1. *If a C^2 function u on a domain Ω is harmonic, then*

$$\int_{\partial\omega} \partial_\nu u\,\mathrm{d}S = 0 \tag{3}$$

for all bounded domains ω with $\overline{\omega} \subset \Omega$ having piecewise C^1 boundary.

Conversely, if (3) holds for every ball $\omega = B(\boldsymbol{x}, r)$ whose closure lies within Ω, then u is harmonic.

Proof. We have already proved the first part. For the converse, (3) and (2) imply that the *average* of Δu over any ball is zero. By letting the radius of the ball $B(\boldsymbol{x}, r)$ tend to zero, we conclude that $\Delta u(\boldsymbol{x}) = 0$. ∎

Definition. The (*radius r*) *spherical average* of a function u at a point $\boldsymbol{x}$ is defined to be

$$\tilde{u}_{\boldsymbol{x}}(r) = \fint_{\partial B(\boldsymbol{x},r)} u\,\mathrm{d}S = \fint_{\mathbb{S}^{n-1}} u(\boldsymbol{x} + r\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere and the "barred" integral signs denote the average:

$$\fint_{\partial B(\boldsymbol{x},r)} u \, \mathrm{d}S = \frac{1}{A_n r^{n-1}} \int_{\partial B(\boldsymbol{x},r)} u \, \mathrm{d}S,$$

where A_n is the area of $\mathbb{S}^{n-1}$. Note that the second integral in the definition of $\tilde{u}_{\boldsymbol{x}}(r)$ makes sense even for $r < 0$; thus, we adopt this as the definition for all real r for which the integrand is defined on $\mathbb{S}^{n-1}$. We see that $\tilde{u}_{\boldsymbol{x}}$ is an *even* function; it is C^k if u is C^k, and $\tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$.

When ω is a ball, we can rewrite (2) in terms of spherical averages: Noting that the volume of the ball $B(\boldsymbol{x}, r)$ is $A_n r^n/n$, we find

$$\fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} = \frac{n}{r} \fint_{B(\boldsymbol{x},r)} \partial_\nu u(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) = \frac{n}{r} \fint_{\mathbb{S}^{n-1}} \partial_r u(\boldsymbol{x} + r\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}),$$

where we can move the r derivative outside the integral, and arrive at

$$\fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} = \frac{n}{r} \tilde{u}'_{\boldsymbol{x}}(r). \tag{4}$$

Along with $\tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$, this implies

Theorem 2 (The mean value property of harmonic functions). *A C^2 function u on a domain Ω is harmonic if and only if $\tilde{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ for all $x \in \Omega$ and all r for which $\overline{B}(\boldsymbol{x}, |r|) \subset \Omega$.*

In general, we say a function u satisfies the *mean value property* if $\tilde{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ whenever $\overline{B}(\boldsymbol{x}, |r|) \subset \Omega$. We hall see below (Theorem 4) that the mean value property characterizes harmonic functions. But first, we collect an easy consequence of (4).

Proposition 3. *For any C^2 function u, we have*

$$\Delta u(\boldsymbol{x}) = n\tilde{u}''_{\boldsymbol{x}}(0).$$

Proof. The function $\tilde{u}_{\boldsymbol{x}}$ is even, so $\tilde{u}'_{\boldsymbol{x}}(0) = 0$. Therefore, letting $r \to 0$ in (4), we arrive at the stated result. ∎

Theorem 4 (The mean value property and regularity). *Assume that a continuous function u satisfies the mean value property on a domain Ω. Then u is infinitely differentiable, and is therefore harmonic. In particular, every harmonic function is infinitely differentiable.*

Proof. This proof may seem long, but only because we use it to develop some tools that have wider applicability.

First, define a *standard mollifier* $\rho\colon \mathbb{R}^n \to \mathbb{R}$. Here is one of many possible definitions:

$$\rho(\boldsymbol{x}) = \begin{cases} ae^{1/(|\boldsymbol{x}|^2-1)}, & |\boldsymbol{x}| < 1 \\ 0, & |\boldsymbol{x}| \geq 1, \end{cases}$$

where $a > 0$ is chosen to ensure that

$$\int_{\mathbb{R}^n} \rho \, \mathrm{d}\boldsymbol{x} = 1.$$

That is one of the defining qualities of a standard mollifier. The others are: That $\rho \geq 0$ everywhere, that it vanishes outside the unit ball, that it is infinitely differentiable, and is radially symmetric – that is, a function of $|\boldsymbol{x}|$ alone.

For any $\delta > 0$ we can squeeze the mollifier to fit inside a ball of radius δ:

$$\rho_\delta(\boldsymbol{x}) = \frac{1}{\delta^n}\rho\Big(\frac{\boldsymbol{x}}{\delta}\Big),$$

so that ρ_δ also has integral 1, but vanishes outside the ball $B(0, \delta)$.

Now we consider the convolution product

$$u * \rho_\delta(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{y})\rho_\delta(\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}.$$

This is defined for all $x \in \Omega$ with a distance less than δ to the complement of Ω. Thus, for any $x \in \Omega$, we can make δ small enough so that $u * \rho_\delta$ is defined at $\boldsymbol{x}$.

Moreover, $u * \rho_\delta$ is infinitely differentiable: This is proved by differentiating with respect to the components of $\boldsymbol{x}$ under the integral sign, as much as you like.

Finally, the mean value property of u and the radial symmetry of ρ_δ combine to ensure that $u(\boldsymbol{x}) = u * \rho(\boldsymbol{x})$ for all $\boldsymbol{x}$ where $u * \rho$ is defined, which is what we were going to prove.

For a detailed argument, write

$$u * \rho(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{x} - \boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}$$

and write the integral in polar form:

$$\begin{aligned} u * \rho(\boldsymbol{x}) &= \int_0^\delta \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{x} - \boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, \mathrm{d}r \\ &= \int_0^\delta \int_{\mathbb{S}^{n-1}} u(\boldsymbol{x} - r\boldsymbol{y})\rho_\delta(r\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, r^{n-1} \, \mathrm{d}r. \end{aligned}$$

Now use the radial symmetry: $\rho_\delta(r\boldsymbol{y})$ is constant for $y \in \mathbb{S}^{n-1}$, so this factor can be moved outside the inner integral. Next, use the mean value property of u:

$$\int_{\mathbb{S}^{n-1}} u(\boldsymbol{x} - r\boldsymbol{y})\rho_\delta(r\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = A_n u(\boldsymbol{x}).$$

But $u(\boldsymbol{x})$ is a constant, which we move outside the outer integral. We are left with

$$u * \rho(\boldsymbol{x}) = u(\boldsymbol{x}) \int_0^\delta r^{n-1} A_n \rho_\delta(r\boldsymbol{y})\,\mathrm{d}r$$

where $\boldsymbol{y}$ is any unit vector. But running the whole calculation in reverse, this time without the u term, reveals that the integral here is merely the integral of ρ_δ, which has the value 1. Thus we are left with $u * \rho(\boldsymbol{x}) = u(\boldsymbol{x})$, as claimed. ∎

The maximum principle

Definition. A C^2 function u is called *subharmonic* if $\Delta u \geq 0$, and *superharmonic* if $\Delta u \leq 0$. Thus it is harmonic if and only if it is both subharmonic and superharmonic. (The reason for the naming will become clear later; see Corollary 7.) Clearly, u is superharmonic if and only if $-u$ is subharmonic.

Theorem 5 (Strong maximum principle)**.** *Assume that $u \in C^2(\Omega)$ is subharmonic in a region $\Omega \subseteq \mathbb{R}^n$. If u has a global maximum in Ω, then u is constant.*

Proof. Let M be the global maximum of u, and put

$$S = \{\boldsymbol{x} \in \Omega \mid u(\boldsymbol{x}) = M\}.$$

Then S is a closed subset of Ω, by the continuity of u. It is also nonempty by assumption.

Consider any $\boldsymbol{x} \in S$. From (4) and the subharmonicity of u, we get $\tilde{u}'_{\boldsymbol{x}}(r) \geq 0$ for $r > 0$. Thus we get $\tilde{u}_{\boldsymbol{x}}(r) \geq \tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x}) = M$ for $r > 0$ (so long as $\overline{B}(\boldsymbol{x}, r) \subset \Omega$). But $u \leq M$ everywhere, and if $u < M$ anywhere on the sphere $\partial B(\boldsymbol{x}, r)$, we would get $\tilde{u}_{\boldsymbol{x}}(r) < 0$. Thus u is contant equal to M on $\partial B(\boldsymbol{x}, r)$ for any small enough r, and therefore, u is constant in some neighbourhood of $\boldsymbol{x}$. This means that S is open.

Since S is an open, closed, and nonempty subset of the connected set Ω, we must have $S = \Omega$, and the proof is complete. ∎

Remark. Obviously, we obtain a strong *minimum* principle for *superharmonic* functions by multiplying by -1. In particular, a non-constant harmonic function cannot attain a minimum or maximum value anywhere in Ω.

Corollary 6 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is subharmonic. Then*

$$\max\{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \overline{\Omega}\,\} = \max\{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \partial\Omega\,\}.$$

In particular, a harmonic function which is continuous on $\overline{\Omega}$ attains its minimum and maximum values on the boundary $\partial\Omega$.

Proof. The weak principle is an obvious consequence of the strong principle. However, it is worth noting that a much more elementary proof exists. Namely, for any $\varepsilon > 0$, let $v(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon|\boldsymbol{x}|^2$, and note that then $\Delta v > 0$. But $\Delta v(\boldsymbol{x}) \le 0$ if $\boldsymbol{x}$ is an interior minimum point for v, so v cannot have any maximum in the interior. Thus for any $\boldsymbol{x} \in \Omega$,

$$u(\boldsymbol{x}) = v(\boldsymbol{x}) - \varepsilon|\boldsymbol{x}|^2 \le \max_{\partial\Omega} v \le \max_{\partial\Omega} u + \varepsilon \max_{x\in\partial\Omega} |\boldsymbol{x}|^2.$$

Now let $\varepsilon \to 0$ to arrive at the conclusion $u(\boldsymbol{x}) \le \max_{\partial\Omega} u$. ∎

Our next result explains the terms *sub*- and *super*harmonic: A subharmonic function is below, and a superharmonic above, a harmonic function given the same boundary data.

Corollary 7. *Assume that Ω is a bounded domain, that $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$, that u is harmonic in Ω, and that $v = u$ on $\partial\Omega$. If v is subharmonic, then $v \le u$ in Ω, while if v is superharmonic, then $v \ge u$ in Ω.*

Proof. Apply the weak maximum principle to $v-u$ if v is subharmonic, or to $u-v$ if v is superharmonic. ∎

Remark. Corollary 7 suggests a strategy for proving existence of a solution to the Dirichlet problem for the Laplace equation: Assume we are trying to find a harmonic function on Ω with the given boundary value g. Consider the pointwise *supremum* of all subharmonic functions which are $\le g$ on $\partial\Omega$, and the pointwise *infimum* of all superharmonic functions which are $\ge g$ on $\partial\Omega$. If the two functions coincide, they should provide a solution to the problem. This is the basis for *Perron's method*, which we will hopefully get a look at later.

The Poisson equation

We now turn our study to the *Poisson equation*:

$$-\Delta u = f \tag{5}$$

where f is a known continuous function. (It *must* be continuous to allow for classical, i.e., C^2, solutions u.)

Referring all the way back to (2), we quickly get the following generalization of Proposition 1:

Proposition 8. *A C^2 function u on a domain Ω solves the Poisson equation* (5) *if and only if*

$$-\int_{\partial\omega} \partial_\nu u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = \int_\omega f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} \tag{6}$$

for all bounded domains ω with $\overline{\omega} \subset \Omega$ having piecewise C^1 boundary. It is sufficient to consider balls $\omega = B(\boldsymbol{x}, r)$.

As an example, we consider a Poisson equation with a radially symmetric right hand side $f(\boldsymbol{x}) = \mathring{f}(|\boldsymbol{x}|)$. We expect to find a radially symmetric solution $u(\boldsymbol{x}) = \mathring{u}(|\boldsymbol{x}|)$. Now (6) with $\omega = B(\boldsymbol{0}, r)$ becomes

$$-A_n r^{n-1}\mathring{u}'(r) = A_n \int_0^r \mathring{f}(s)s^{n-1}\,\mathrm{d}s.$$

Taking the derivative and rearraing turns this into the ODE

$$-\frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\big(r^{n-1}\mathring{u}'(r)\big) = \mathring{f}(r).$$

A direct calculation reveals that indeed,

$$\Delta\mathring{u}(|\boldsymbol{x}|) = \mathring{u}''(|\boldsymbol{x}|) + \frac{n-1}{|\boldsymbol{x}|}\mathring{u}'(|\boldsymbol{x}|) = \frac{1}{r^{n-1}}\frac{\mathrm{d}}{\mathrm{d}r}\big(r^{n-1}\mathring{u}'(r)\big)\bigg|_{r=\boldsymbol{x}}$$

so a solution to the above ODE will in fact solve the Poisson equation in the radially symmetric case.

Consider now the case where $\mathring{f}(r) = 0$ when $r > R$. Then for $r > R$,

$$r^{n-1}\mathring{u}'(r) = \int_0^R \mathring{f}(s)s^{n-1}\,\mathrm{d}s = \frac{1}{A_n}\int_{B(\boldsymbol{0},R)} f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} =: \frac{m}{A_n}.$$

Accordingly, after integrating, we define the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} \Phi(\boldsymbol{x}) &= \frac{-\ln(|\boldsymbol{x}|)}{2\pi} && \text{for } n = 2, \\ \Phi(\boldsymbol{x}) &= \frac{1}{(n-2)A_n|\boldsymbol{x}|^{n-2}} && \text{for } n \geq 3. \end{aligned}$$

Then Φ is harmonic for $\boldsymbol{x} \neq \boldsymbol{0}$, and we find that our radially symmetric solution satisfies

$$u(\boldsymbol{x}) = m\Phi(\boldsymbol{x}) \quad \text{for } |\boldsymbol{x}| > R \qquad \text{where } m = \int_{B(\boldsymbol{0},R)} f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x}$$

– plus an integration constant, which we have thrown away. For $n \geq 3$, this gives the unique radially symmetric solution which vanishes at infinity; for $n = 2$, however, no particular value of the dropped constant of integration distinguishes itself.

Now, assume that $m = 1$, and replace $f(\boldsymbol{x})$ by $e^{-n}f(\boldsymbol{x}/\varepsilon)$, and letting $\varepsilon \to 0$. The resulting solution u will converge pointwise to Φ (except at $\boldsymbol{x} = \boldsymbol{0}$), while f becomes a Dirac δ in the limit. Thus we are tempted to conclude that

$$-\Delta\Phi = \delta$$

This is indeed true, but we first need to get into the theory of distributions in order to understand the rigorous meaning of the above equation.

We call Φ the *fundamental solution* for the Poisson equation. Even lacking the abstract theory, we can use it to solve the general Poisson equation.

Proposed solution to the Poisson equation:

$$u(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y})f(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} \tag{7}$$

We expect this to solve the equation because formally, we get

$$\Delta u = \Delta(\Phi * f) = (\Delta\Phi) * f = \delta * f = f.$$

However, this calculation is hard to justify. We can get there via a detour, if f is sufficiently regular:

Theorem 9. *Assume that $f \in C_c^2(\mathbb{R}^n)$. Then the function u given by (7) is a solution to the Poisson equation (5).*

Proof. To simplify, thanks to translation invariance, we only need to prove that $-\Delta(\Phi * f)(\mathbf{0}) = f(\mathbf{0})$. (For any $\boldsymbol{x}_0 \in \mathbb{R}^n$, put $\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}_0 + \boldsymbol{x})$. Then $\Phi * \tilde{f}(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}_+ + \boldsymbol{x})$, so $-\Delta(\Phi * \tilde{f})(\mathbf{0}) = \tilde{f}(\mathbf{0})$ implies $-\Delta(\Phi * f)(\boldsymbol{x}) = f(\boldsymbol{x})$.)

First, we note that

$$\int_{\partial B(\mathbf{0},r)} \Phi \, \mathrm{d}^x \boldsymbol{x} = \begin{cases} -r \ln r & \text{for } n = 2, \\ r/(n-2) & \text{for } n \geq 3. \end{cases}$$

Now integrating with respect to r, we conclude that Φ is integrable (meaning the integral of $|\Phi|$ is finite) over $B(\mathbf{0}, r)$, and hence over any bounded subset of $\mathbb{R}^n$. From this, we conclude that not only is

$$u(\boldsymbol{x}) = \Phi * f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y}$$

well defined, but u is C^2 as well, and in fact

$$\Delta u = \Phi * \Delta f.$$

This is easy if f has compact support; it is also true if f and its derivatives up to second order vanish sufficiently fast at infinity, but we are not going to bother with this refinement.

In particular,

$$\Delta u(\mathbf{0}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y}) \, \Delta f(-\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} = \int_{\mathbb{R}^n} \Phi(\boldsymbol{y}) \, \Delta f(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y}$$

thanks the symmetry of Φ. (Not essential, but it's one less minus sign to track.) We want to use Green's second identity to move the Laplacian to Φ instead. But then we need to restrict attention to a bounded region, and we need to avoid the singularity of Φ at the origin. (It is this singularity that causes the answer to be non-zero, after all.) Pick R sufficiently large so $f(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \geq R$, so that integrating over $B(\mathbf{0}, R)$ does not change the integral. Then, thanks to the integrabilty of Φ near the origin, we find

$$\Delta u(\mathbf{0}) = \lim_{\varepsilon \to 0} \int_{B(\mathbf{0},R) \setminus B(\mathbf{0},\varepsilon)} \Phi(\boldsymbol{y}) \, \Delta f(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y}.$$

This integral can be transformed by Green's second identity, resulting in

$$\Delta u(\mathbf{0}) = \lim_{\varepsilon \to 0} \Big(\int_{B(\mathbf{0},R) \setminus B(\mathbf{0},\varepsilon)} \Delta\Phi(\boldsymbol{y}) f(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} + \int_{\partial(B(\mathbf{0},R) \setminus B(\mathbf{0},\varepsilon))} (\Phi \partial_\nu f - f \partial_\nu \Phi) \, \mathrm{d}S \Big).$$

The first integral vanishes because $\Delta\Phi = 0$, and in the second integral we can ignore the outer boundary $\partial B(\mathbf{0}, R)$ because $f = \partial_\nu f = 0$ there. Finally, the normal vector ν points *inward* on the inner boundary $\partial B(\mathbf{0},\varepsilon)$, so we get a sign change when we consider the outward point normal instead. Thus we have

$$\Delta u(\mathbf{0}) = \lim_{\varepsilon\to 0} \int_{\partial B(\mathbf{0},\varepsilon)} (f\partial_\nu \Phi - \Phi\partial_\nu f)\,\mathrm{d}S.$$

Here, the integral of $\Phi\partial_\nu f$ vanishes in the limit as $\varepsilon \to 0$, since $\partial_\nu f$ is bounded. And on $\partial B(\mathbf{0},\varepsilon)$, we find $\partial_\nu \Phi = -1/(A_n r^{n-1})$, so

$$\Delta u(\mathbf{0}) = -\lim_{\varepsilon\to 0} \frac{1}{A_n r^{n-1}} \int_{\partial B(\mathbf{0},\varepsilon)} f\,\mathrm{d}S = -f(\mathbf{0}),$$

and the proof is complete. ∎

Bounded domains and Green's function. Let Ω be a bounded domain with piecewise C^1 boundary. The weak maximum principle (even without the "nice" boundary) immediately shows that the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{DP}$$

has at most one solution for any given data f and g.

Our goal in this section is to generalize the representation formula (7) to this setting. We first concentrate on homogeneous boundary data, i.e., $g = 0$. It will turn out that we get the case for $g \neq 0$ "for free".

First, rewrite (7) to read

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}.$$

A similar formula for solutions on Ω might look like

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^n} G_{\boldsymbol{y}}(\boldsymbol{x}) f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}$$

instead. To ensure that this satisfies $u = 0$ on $\partial\Omega$, we want $G_{\boldsymbol{y}}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial\Omega$. Apart from that, we want $G_{\boldsymbol{y}}(\boldsymbol{x})$ (as a functin of $\boldsymbol{x}$) to be as much "like" $\Phi(\boldsymbol{x}-\boldsymbol{y})$ as possible. To make that precise:

Definition. Assume that, for each $\boldsymbol{y} \in \Omega$, there exists a function $H_{\boldsymbol{y}} \in C(\overline{\Omega})$ which is harmonic in Ω and satisfies

$$H_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y}) \quad \text{for } \boldsymbol{x} \in \partial\Omega.$$

By the maximum principle, this function is unique. Then the function $G_{\boldsymbol{y}}$ defined by

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y}) - H_{\boldsymbol{y}}(\boldsymbol{x})$$

is called the *Green's function* associated with Ω. Clearly, this function is continuous on $\overline{\Omega} \setminus \{\boldsymbol{y}\}$ and harmonic on $\Omega \setminus \{\boldsymbol{y}\}$, and it vanishes on $\partial\Omega$.

To discover how this helps us write a solution formula for the Dirichlet problem, we concentrate first on the Φ part.

For simplicity, we temporarily put $\boldsymbol{y} = \mathbf{0}$, assuming that $\mathbf{0} \in \Omega$. If $\varepsilon > 0$ is small enough, $\overline{B}(\mathbf{0},\varepsilon) \subset \Omega$. Put $\Omega_\varepsilon = \Omega \setminus \overline{B}(\mathbf{0},\varepsilon)$, and apply Green's second identity to Φ and an arbitrary function $u \in C^2(\overline{\Omega})$:

$$\begin{aligned}\int_{\Omega_\varepsilon} \big(u\,\Delta\Phi - \Phi\,\Delta u\big)\,\mathrm{d}^n\boldsymbol{x} &= \int_{\partial\Omega_\varepsilon} \big(u\,\partial_\nu\Phi - \Phi\,\partial_\nu u\big)\,\mathrm{d}S \\ &= \int_{\partial\Omega} \big(u\,\partial_\nu\Phi - \Phi\,\partial_\nu u\big)\,\mathrm{d}S - \int_{\partial B(\mathbf{0},\varepsilon)} \big(u\,\partial_\nu\Phi - \Phi\,\partial_\nu u\big)\,\mathrm{d}S,\end{aligned}$$

where the minus sign in front of the last integral is due to the direction of the normal vector $\boldsymbol{\nu}$ on $\partial B(\mathbf{0},\varepsilon)$ pointing out of the ball, whereas the normal vector on that part of $\partial\Omega_\varepsilon$ points into the ball.

Now let $\varepsilon \to 0$. The red term on the left is already zero, while the boundary integral of the red term on the right will vanish in the limit. (For $|\partial_\nu u| \le |\nabla u|$, which is bounded, and the integral of Φ converges to zero.) Finally, the integral of the green term will converge to $-u(\mathbf{0})$, so we end up with the representation formula

$$u(\boldsymbol{y}) = -\int_{\Omega} \Phi\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega} \big(u\,\partial_\nu\Phi - \Phi\,\partial_\nu u\big)\,\mathrm{d}S, \qquad \boldsymbol{y} \in \Omega. \tag{8}$$

(We proved it only for $\boldsymbol{y} = \mathbf{0}$, but translation invariance ensures that the result generalizes. Or you could rerun the proof with $\Phi(\boldsymbol{x})$ replaced by $\Phi_{\boldsymbol{y}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{y})$ and $B(\mathbf{0},\varepsilon)$ by $B(\boldsymbol{y},\varepsilon)$.)

Repeat the same calculation with Φ replaced by $H_{\boldsymbol{y}}$. This is much easier, since $H_{\boldsymbol{y}}$ is harmonic we do not need to remove a ball: Green's second identity immediately yields

$$\int_{\Omega} \big(u\;\Delta H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\,\Delta u\big)\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\Omega} \big(u\,\partial_\nu H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\partial_\nu u\big)\,\mathrm{d}S,$$

where again the red term vanishes. Rearranging this into

$$\int_{\Omega} H_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} + \int_{\partial\Omega} \big(u\,\partial_\nu H_{\boldsymbol{y}} - H_{\boldsymbol{y}}\partial_\nu u\big)\,\mathrm{d}S = 0$$

and adding it to the right hand side of (8), we obtain

$$u(\boldsymbol{y}) = -\int_{\Omega} G_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega} \big(u\,\partial_\nu G_{\boldsymbol{y}} - G_{\boldsymbol{y}}\,\partial_\nu u\big)\,\mathrm{d}S$$

where the red term is zero by construction, so we finally have

$$u(\boldsymbol{y}) = -\int_{\Omega} G_{\boldsymbol{y}}\,\Delta u\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega} u\,\partial_\nu G_{\boldsymbol{y}}\,\mathrm{d}S, \qquad \boldsymbol{y}\in\Omega.$$

We have proved

Theorem 10. *If $\Omega \subset \mathbb{R}^n$ is a bounded region with C^1 boundary, and if Ω admits a Green's function G, then the solution $u \in C^2(\overline{\Omega})$ of* (DP)*, if it exists, is given by*

$$u(\boldsymbol{y}) = \int_{\Omega} G_{\boldsymbol{y}}(\boldsymbol{x})\,f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} - \int_{\partial\Omega} \partial_\nu G_{\boldsymbol{y}}(\boldsymbol{x})\,g(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}), \qquad \boldsymbol{y}\in\Omega. \tag{9}$$

Notice the careful wording: We have not shown that the Dirichlet problem has a solution. We have only shown what it must be, if it solves the problem.

We now show that Green's function is (usually) symmetric.

Proposition 11. *Assume that Ω is a bounded region with piecewice C^1 boundary, for which the Dirichlet problem* (DP) *always has a solution for any given continuous data. In particular, this implies the existence of a Green's function. Then the Green's function G for this region satisfies $G_{\boldsymbol{y}}(\boldsymbol{x}) = G_{\boldsymbol{x}}(\boldsymbol{y})$, whenever $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.*

Proof. Let $f, g \in C(\overline{\Omega})$, and let u, v solve $-\Delta u = f$ and $-\Delta v = g$ in Ω, $u = v = 0$ on $\partial\Omega$. Thus

$$u(\boldsymbol{y}) = \int_{\Omega} G_{\boldsymbol{y}}(\boldsymbol{x})\,f(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x}, \quad v(\boldsymbol{y}) = \int_{\Omega} G_{\boldsymbol{y}}(\boldsymbol{x})\,g(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x}.$$

Since u and v vanish on $\partial\Omega$, Green's second identity applied to u and v implies

$$\int_{\Omega} u(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = \int_{\Omega} v(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y}.$$

Plugging in the above formulas for u and v, swapping the integration variables $\boldsymbol{x}$ and $\boldsymbol{y}$ in one of the integrals, and rearranging, we get

$$\int_{\Omega}\int_{\Omega} \big(G_{\boldsymbol{y}}(\boldsymbol{x}) - G_{\boldsymbol{x}}(\boldsymbol{y})\big)f(\boldsymbol{x})g(\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{y} = 0,$$

from which the desired conclusion quickly follows.

In more detail, let ρ be a standard mollifier, and put $\rho_\varepsilon(\boldsymbol{x}) = \varepsilon^{-n}\rho(\boldsymbol{x}/\varepsilon)$. If $\boldsymbol{x}_0, \boldsymbol{y}_y \in \Omega$ and $\boldsymbol{x}_0 \neq \boldsymbol{y}_y$, put $f(\boldsymbol{x}) = \rho_\varepsilon(\boldsymbol{x} - \boldsymbol{x}_0)$ and $g(\boldsymbol{y}) = \rho_\varepsilon(\boldsymbol{y} - \boldsymbol{y}_0)$. Inserting these into the double integral and letting $\varepsilon \to 0$, we get $G_{\boldsymbol{y}_0}(\boldsymbol{x}_0) - G_{\boldsymbol{x}_0}(\boldsymbol{y}_0)$ in the limit, so that $G_{\boldsymbol{y}_0}(\boldsymbol{x}_0) = G_{\boldsymbol{x}_0}(\boldsymbol{y}_0)$. ∎

Remark. It may be interesting to study the above proof from an elementary linear algebra point of view. The last part resembles the proof that, if A is a quadratic matrix satisfying $y^T A x = x^t A y$ for all vectors x and y, then A is symmetric. (Just take $x = e_i$ and $y = e_j$.) The first part resembles the proof that the inverse of an invertible symmetric matrix is symmetric. Think of the "matrix" $G_{\boldsymbol{x}}(\boldsymbol{y})$ (where $\boldsymbol{x}$ and $\boldsymbol{y}$ play the rôles of matrix indices) as the inverse of the operator $-\Delta$. And the symmetry of this operator is just the identity $\int_\Omega u\,\Delta v = \int_\Omega v\,\Delta u$. (Green's second identity, when both functions vanish on the boundary.)

Green's function for balls. Here we compute Green's function for the open unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. I will drop some exponents and write $\mathbb{B}$ instead of $\mathbb{B}^m$ and $\mathbb{S} = \partial\mathbb{B}$ instead of $\mathbb{S}^{n-1}$.

If $\boldsymbol{y} \in \mathbb{B}$, we need to find a harmonic function on $\mathbb{B}$ with the same values as $\Phi(\boldsymbol{x} - \boldsymbol{y})$ for $\boldsymbol{x} \in \mathbb{S}$. The trick is to "put a charge" at a suitable point $\boldsymbol{w}$ outside $\mathbb{B}$. It turns out that $\boldsymbol{w} = \boldsymbol{y}/|\boldsymbol{y}|^2$ does the trick. Since $|\boldsymbol{x}| = 1$, we find

$$|\boldsymbol{x} - \boldsymbol{y}|^2 = 1 + |\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}$$

and

$$|\boldsymbol{x} - \boldsymbol{w}|^2 = 1 + |\boldsymbol{w}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{w} = 1 + \frac{1}{|\boldsymbol{y}|^2} - 2\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{y}|^2} = \frac{|\boldsymbol{y} - \boldsymbol{x}|^2}{|\boldsymbol{y}|^2},$$

so that

$$|\boldsymbol{w} - \boldsymbol{x}| = \frac{|\boldsymbol{y} - \boldsymbol{x}|}{|\boldsymbol{y}|}.$$

For $n \geq 3$, this gives

$$\Phi(\boldsymbol{w} - \boldsymbol{x}) = \frac{1}{(n-2)A_n|\boldsymbol{w} - \boldsymbol{x}|^{n-2}} = |\boldsymbol{y}|^{n-2}\Phi(\boldsymbol{y} - \boldsymbol{x}),$$

so we should put

$$H_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{\Phi(\boldsymbol{w} - \boldsymbol{x})}{|\boldsymbol{y}|^{n-2}} = \frac{1}{(n-2)A_n|\boldsymbol{y}|^{n-2}|\boldsymbol{w} - \boldsymbol{x}|^{n-2}}$$

where we note that

$$|\boldsymbol{y}|^2|\boldsymbol{w} - \boldsymbol{x}|^2 = |\boldsymbol{y}|^2(|\boldsymbol{w}|^2 + |\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{w}) = 1 + |\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y},$$

so we get

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{1}{(n-2)A_n}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{n-2}} - \frac{1}{(1+|\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y})^{(n-2)/2}}\right) \quad (n \geq 3)$$

which is indeed symmetric in $\boldsymbol{x}$, $\boldsymbol{y}$ as expected.

Next, the same computation for $n = 2$: We find (when $\boldsymbol{x} \in \mathbb{S}$)

$$\Phi(\boldsymbol{w}-\boldsymbol{x}) = -\frac{\ln|\boldsymbol{w}-\boldsymbol{x}|}{2\pi} = \frac{\ln|\boldsymbol{y}|}{2\pi} + \Phi(\boldsymbol{y}-\boldsymbol{x}),$$

so we put

$$H_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln|\boldsymbol{y}|}{2\pi} + \Phi(\boldsymbol{w}-\boldsymbol{x}) = -\frac{\ln(|\boldsymbol{y}||\boldsymbol{w}-\boldsymbol{x}|)}{2\pi}$$

and get

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln|\boldsymbol{y}-\boldsymbol{x}|}{2\pi} + \frac{\ln(1+|\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y})}{4\pi} \quad (n = 2)$$

which is again symmetric. We can also write this on the form

$$G_{\boldsymbol{y}}(\boldsymbol{x}) = -\frac{\ln(|\boldsymbol{x}|^2+|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y})}{4\pi} + \frac{\ln(1+|\boldsymbol{x}|^2|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y})}{4\pi} \quad (n = 2)$$

When solving the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$, we need the normal derivative $\partial_\nu G_{\boldsymbol{y}}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{S}$ which is simply the derivative of $G_{\boldsymbol{y}}(t\boldsymbol{x})$ taken at $t = 1$, resulting in

$$\partial_\nu G_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{-2+2\boldsymbol{x}\cdot\boldsymbol{y}+2|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}}{4\pi(1+|\boldsymbol{y}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y})} = \frac{|\boldsymbol{y}|^2-1}{2\pi(1+|\boldsymbol{y}|^2-2\boldsymbol{x}\cdot\boldsymbol{y})}.$$

If we put $r = |\boldsymbol{y}|$ and let θ be the angle between the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we get

$$-\partial_\nu G_{\boldsymbol{y}}(\boldsymbol{x}) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}$$

which is known as the *Poisson kernel*. Referring back to (9), the solution to the Dirichlet problem $-\Delta u = 0$ on $\mathbb{D}$ with $u(\cos\theta, \sin\theta) = g(\theta)$ should then be given by

$$u(r\cos\varphi, r\sin\varphi) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{g(\varphi-\theta)}{1+r^2-2r\cos\theta}\,\mathrm{d}\theta$$

This is known as *Poisson's integral formula*. It was originally derived using separation of variables and a Fourier analysis:

For this, we may note that the Laplace operator in polar coordinates takes the form

$$\Delta u = r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta}.$$

We look for harmonic functions of the form $u = R(r)\Theta(\theta)$, resulting in $r^{-1}(rR')'\Theta + r^{-2}R\Theta'' = 0$. After separating the variables, we are left with $\Theta'' = -\lambda\Theta$ and $(rR')' = \lambda r^{-1}R$. Since Θ must be 2π-periodic, we must put $\lambda = n^2$ for an integer n, so the R equation is $r(rR')' = n^2R$. We try $R = r^k$, and get $rR' = kr^k$, $r(rR')' = k^2r^k$, with the non-trivial solutions $k = \pm n$. Rejecting solutions with negative k (which have a singularity at $r = 0$), we are therefore left with $R = r^{|n|}$. Thus we are led to look for solutions of the form

$$u(r\cos\varphi, r\sin\varphi) = \sum_{n=-\infty}^{\infty} c_n r^{|n|}e^{in\varphi}.$$

Matching this to $g(\theta)$ at $r = 1$, we should have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}\,d\theta,$$

and therefore

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} g(\theta)r^{|n|}e^{in(\varphi-\theta)}\,d\theta. = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)P(r,\varphi-\theta)\,d\theta,$$

where

$$P(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}.$$

Here we note that

$$\sum_{k=0}^{\infty} r^k e^{\pm ik\theta} = \frac{1}{1 - re^{\pm ik\theta}},$$

so

$$\begin{aligned} P(r,\theta) &= \frac{1}{2\pi}\Big(\frac{1}{1-re^{ik\theta}} + \frac{1}{1-re^{-ik\theta}} - 1\Big) \\ &= \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}, \end{aligned}$$

which we recognize as the Poisson kernel introduced above.