Harmonicfunctionology

Harald Hanche-Olsen

The Laplace operator on $\mathbb{R}^n$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

plays a rôle in the wave and heat equations, but even more fundamentally, in the *Laplace equation*

$$\Delta u = 0 \tag{1}$$

and the *Poisson equation* $-\Delta u = f$ where f is a given function.

A C^2 solution of (1) is called *harmonic*. (Later, we will find that harmonic functions are in fact infinitely differentiable.)

Much is gained by the trivial observation that $\Delta u = \nabla \cdot \nabla u$ together with various applications of the divergence theorem or two of its corollaries, Green's first and second identities.

Let's get started by simply integrating Δu over a bounded domain ω with piecewise C^1 boundary:

$$\int_\omega \Delta u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_\omega \nabla \cdot \nabla u(\boldsymbol{x})\,\mathrm{d}^n\boldsymbol{x} = \int_{\partial\omega} \partial_\nu u(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}). \tag{2}$$

This immediately proves

Proposition 1. *If a C^2 function u on a domain Ω is harmonic, then*

$$\int_{\partial\omega} \partial_\nu u\,\mathrm{d}S = 0 \tag{3}$$

for all bounded domains $\omega \subseteq \Omega$ with piecewise C^1 boundary.

Conversely, if (3) holds for every ball $\omega = B(\boldsymbol{x}, r)$ whose closure lies within Ω, then u is harmonic.

Proof. We have already proved the first part. For the converse, (3) and (2) imply that the *average* of Δu over any ball is zero. By letting the radius of the ball $B(\boldsymbol{x}, r)$ tend to zero, we conclude that $\Delta u(\boldsymbol{x}) = 0$. ∎

Definition. The *(radius r) spherical average* of a function u at a point $\boldsymbol{x}$ is defined to be

$$\tilde{u}_{\boldsymbol{x}}(r) = \fint_{\partial B(\boldsymbol{x},r)} u\,\mathrm{d}S = \fint_{\mathbb{S}^{n-1}} u(\boldsymbol{x} + r\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}),$$

where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere and the "barred" integral signs denote the average:

$$\fint_{\partial B(\boldsymbol{x},r)} u \, \mathrm{d}S = \frac{1}{A_n r^{n-1}} \int_{\partial B(\boldsymbol{x},r)} u \, \mathrm{d}S,$$

where A_n is the area of $\mathbb{S}^{n-1}$. Note that the second integral in the definition of $\tilde{u}_{\boldsymbol{x}}(r)$ makes sense even for $r < 0$; thus, we adopt this as the definition for all real r for which the integrand is defined on $\mathbb{S}^{n-1}$. We see that $\tilde{u}_{\boldsymbol{x}}$ is an *even* function; it is C^k if u is C^k, and $\tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$.

When ω is a ball, we can rewrite (2) in terms of spherical averages: Noting that the volume of the ball $B(\boldsymbol{x}, r)$ is $A_n r^n/n$, we find

$$\fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} = \frac{n}{r} \fint_{B(\boldsymbol{x},r)} \partial_\nu u(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) = \frac{n}{r} \fint_{\mathbb{S}^{n-1}} \partial_r u(\boldsymbol{x} + r\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}),$$

where we can move the r derivative outside the integral, and arrive at

$$\fint_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y}) \, \mathrm{d}^n \boldsymbol{y} = \frac{n}{r} \tilde{u}'_{\boldsymbol{x}}(r). \tag{4}$$

Along with $\tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x})$, this implies

Theorem 2 (The mean value property of harmonic functions). *A C^2 function u on a domain Ω is harmonic if and only if $\tilde{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ for all $x \in \Omega$ and all r for which $\overline{B}(\boldsymbol{x}, |r|) \subset \Omega$.*

In general, we say a function u satisfies the *mean value property* if $\tilde{u}_{\boldsymbol{x}}(r) = u(\boldsymbol{x})$ whenever $\overline{B}(\boldsymbol{x}, |r|) \subset \Omega$. We hall see below (Theorem 4) that the mean value property characterizes harmonic functions. But first, we collect an easy consequence of (4).

Proposition 3. *For any C^2 function u, we have*

$$\Delta u(\boldsymbol{x}) = n\tilde{u}''_{\boldsymbol{x}}(0).$$

Proof. The function $\tilde{u}_{\boldsymbol{x}}$ is even, so $\tilde{u}'_{\boldsymbol{x}}(0) = 0$. Therefore, letting $r \to 0$ in (4), we arrive at the stated result. ∎

Theorem 4 (The mean value property and regularity). *Assume that a continuous function u satisfies the mean value property on a domain Ω. Then u is infinitely differentiable, and is therefore harmonic. In particular, every harmonic function is infinitely differentiable.*

Proof. This proof may seem long, but only because we use it to develop some tools that have wider applicability.

First, define a *standard mollifier* $\rho\colon \mathbb{R}^n \to \mathbb{R}$. Here is one of many possible definitions:

$$\rho(\boldsymbol{x}) = \begin{cases} a e^{1/(|\boldsymbol{x}|^2-1)}, & |\boldsymbol{x}| < 1 \\ 0, & |\boldsymbol{x}| \geq 1, \end{cases}$$

where $a > 0$ is chosen to ensure that

$$\int_{\mathbb{R}^n} \rho \, \mathrm{d}\boldsymbol{x} = 1.$$

That is one of the defining qualities of a standard mollifier. The others are: That $\rho \geq 0$ everywhere, that it vanishes outside the unit ball, that it is infinitely differentiable, and is radially symmetric – that is, a function of $|\boldsymbol{x}|$ alone.

For any $\delta > 0$ we can squeeze the mollifier to fit inside a ball of radius δ:

$$\rho_\delta(\boldsymbol{x}) = \frac{1}{\delta^n}\rho\Big(\frac{\boldsymbol{x}}{\delta}\Big),$$

so that ρ_δ also has integral 1, but vanishes outside the ball $B(0, \delta)$.

Now we consider the convolution product

$$u * \rho_\delta(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{y})\rho_\delta(\boldsymbol{x}-\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}.$$

This is defined for all $x \in \Omega$ with a distance less than δ to the complement of Ω. Thus, for any $x \in \Omega$, we can make δ small enough so that $u * \rho_\delta$ is defined at $\boldsymbol{x}$.

Moreover, $u * \rho_\delta$ is inifinetly differentiable: This is proved by differentiating with respect to the components of $\boldsymbol{x}$ under the integral sign, as much as you like.

Finally, the mean value property of u and the radial symmetry of ρ_δ combine to ensure that $u(\boldsymbol{x}) = u * \rho(\boldsymbol{x})$ for all $\boldsymbol{x}$ where $u * \rho$ is defined, which is what we were going to prove.

For a detailed argument, write

$$u * \rho(\boldsymbol{x}) = \int_{\mathbb{R}^n} u(\boldsymbol{x}-\boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}^n\boldsymbol{y}$$

and write the integral in polar form:

$$\begin{aligned} u * \rho(\boldsymbol{x}) &= \int_0^\delta \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{x}-\boldsymbol{y})\rho_\delta(\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, \mathrm{d}r \\ &= \int_0^\delta r^{n-1} \int_{\mathbb{S}^{n-1}} u(\boldsymbol{x}-r\boldsymbol{y})\rho_\delta(r\boldsymbol{y}) \, \mathrm{d}S(\boldsymbol{y}) \, \mathrm{d}r. \end{aligned}$$

Now use the radial symmetry: $\rho_\delta(r\boldsymbol{y})$ is constant for $y \in \mathbb{S}^{n-1}$, so this factor can be moved outside the inner integral. Next, use the mean value property of u:

$$\int_{\mathbb{S}^{n-1}} u(\boldsymbol{x} - r\boldsymbol{y})\rho_\delta(r\boldsymbol{y})\,\mathrm{d}^n\boldsymbol{y} = A_n u(\boldsymbol{x}).$$

But $u(\boldsymbol{x})$ is a constant, which we move outside the outer integral. We are left with

$$u * \rho(\boldsymbol{x}) = u(\boldsymbol{x}) \int_0^\delta r^{n-1} A_n \rho_\delta(r\boldsymbol{y})\,\mathrm{d}r$$

where $\boldsymbol{y}$ is any unit vector. But running the whole calculation in reverse, this time without the u term, reveals that the integral here is merely the integral of ρ_δ, which has the value 1. Thus we are left with $u * \rho(\boldsymbol{x}) = u(\boldsymbol{x})$, as claimed. ∎

The maximum principle

Definition. A C^2 function u is called *subharmonic* if $\Delta u \geq 0$, and *superharmonic* if $\Delta u \leq 0$. Thus it is harmonic if and only if it is both subharmonic and superharmonic. (The reason for the naming will become clear later.) Obviously, u is superharmonic if and only if $-u$ is subharmonic.

Theorem 5 (Strong maximum principle). *Assume that $u \in C^2(\Omega)$ is subharmonic in a region $\Omega \subseteq \mathbb{R}^n$. If u has a global maximum in Ω, then u is constant.*

Proof. Let M be the global maximum of u, and put

$$S = \{\boldsymbol{x} \in \Omega \mid u(\boldsymbol{x}) = M\}.$$

Then S is a closed subset of Ω, by the continuity of u. It is also nonempty by assumption.

Consider any $\boldsymbol{x} \in S$. From (4) and the subharmonicity of u, we get $\tilde{u}'_{\boldsymbol{x}}(r) \geq 0$ for $r > 0$. Thus we get $\tilde{u}_{\boldsymbol{x}}(r) \geq \tilde{u}_{\boldsymbol{x}}(0) = u(\boldsymbol{x}) = M$ for $r > 0$ (so long as $\overline{B}(\boldsymbol{x}, r) \subset \Omega$). But $u \leq M$ everywhere, and if $u < M$ anywhere on the sphere $\partial B(\boldsymbol{x}, r)$, we would get $\tilde{u}_{\boldsymbol{x}}(r) < 0$. Thus u is contant equal to M on $\partial B(\boldsymbol{x}, r)$ for any small enough r, and therefore, u is constant in some neighbourhood of $\boldsymbol{x}$. This means that S is open.

Since S is an open, closed, and nonempty subset of the connected set Ω, we must have $S = \Omega$, and the proof is complete. ∎

Remark. Obviously, we obtain a strong *minimum* principle for *superharmonic* functions by multiplying by -1. In particular, a non-constant harmonic function cannot attain a minimum or maximum value anywhere in Ω.

Corollary 6 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is subharmonic. Then*

$$\max\{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \overline{\Omega}\,\} = \max\{\, u(\boldsymbol{x}) \mid \boldsymbol{x} \in \partial\Omega\,\}.$$

In particular, a harmonic function which is continuous on $\overline{\Omega}$ attains its minimum and maximum values on the boundary $\partial\Omega$.

Proof. The weak principle is an obvious consequence of the strong principle. However, it is worth noting that a much more elementary proof exists. Namely, for any $\varepsilon > 0$, let $v(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon|\boldsymbol{x}|^2$, and note that then $\Delta v > 0$. But $\Delta v(\boldsymbol{x}) \le 0$ if $\boldsymbol{x}$ is an interior minimum point for v, so v cannot have any maximum in the interior. Thus for any $\boldsymbol{x} \in \Omega$,

$$u(\boldsymbol{x}) = v(\boldsymbol{x}) - \varepsilon|\boldsymbol{x}|^2 \le \max_{\partial\Omega} v \le \max_{\partial\Omega} u + \varepsilon \max_{x\in\partial\Omega} |\boldsymbol{x}|^2.$$

Now let $\varepsilon \to 0$ to arrive at the conclusion $u(\boldsymbol{x}) \le \max_{\partial\Omega} u$. ∎

Our next result explains the terms *sub*- and *super*harmonic: A subharmonic function is below, and a superharmonic above, a harmonic function given the same boundary data.

Corollary 7. *Assume that Ω is a bounded domain, that $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$, that u is harmonic in Ω, and that $v = u$ on $\partial\Omega$. If v is subharmonic, then $v \le u$ in Ω, while if v is superharmonic, then $v \ge u$ in Ω.*

Proof. Apply the weak maximum principle to $v - u$ if v is subharmonic, or to $u - v$ if v is superharmonic. ∎

Remark. Corollary 7 suggests a strategy for proving existence of a solution to the Dirichlet problem for the Laplace equation: Assume we are trying to find a harmonic function on Ω with the given boundary value g. Consider the pointwise *supremum* of all subharmonic functions which are $\le g$ on $\partial\Omega$, and the pointwise *infimum* of all superharmonic functions which are $\ge g$ on $\partial\Omega$. If the two functions coincide, they should provide a solution to the problem. This is the basis for *Perron's method*, which we will hopefully get a look at later.