

## Harmonicfunctionology

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The Laplace operator on  $\mathbb{R}^n$

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

plays a rôle in the wave and heat equations, but even more fundamentally, in the *Laplace equation*

$$\Delta u = 0 \quad (1)$$

and the *Poisson equation*  $-\Delta u = f$  where  $f$  is a given function.

A  $C^2$  solution of (1) is called *harmonic*. (Later, we will find that harmonic functions are in fact infinitely differentiable.)

Much is gained from the trivial observation that  $\Delta u = \nabla \cdot \nabla u$  together with various applications of the divergence theorem or two of its corollaries, Green's first and second identities.

Let's get started by simply integrating  $\Delta u$  over a bounded domain  $\omega$  with piecewise  $C^1$  boundary:

$$\int_{\omega} \Delta u(\mathbf{x}) \, d^n \mathbf{x} = \int_{\omega} \nabla \cdot \nabla u(\mathbf{x}) \, d^n \mathbf{x} = \int_{\partial \omega} \partial_{\nu} u(\mathbf{x}) \, dS(\mathbf{x}). \quad (2)$$

This immediately proves

**Proposition 1.** *If a  $C^2$  function  $u$  on a domain  $\Omega$  is harmonic, then*

$$\int_{\partial \omega} \partial_{\nu} u \, dS = 0 \quad (3)$$

for all bounded domains  $\omega \subseteq \Omega$  with piecewise  $C^1$  boundary.

Conversely, if (3) holds for every ball  $\omega = B(\mathbf{x}, r)$  whose closure lies within  $\Omega$ , then  $u$  is harmonic.

*Proof.* We have already proved the first part. For the converse, (3) and (2) imply that the *average* of  $\Delta u$  over any ball is zero. By letting the radius of the ball  $B(\mathbf{x}, r)$  tend to zero, we conclude that  $\Delta u(\mathbf{x}) = 0$ . ■

**Definition.** The (*radius  $r$* ) *spherical average* of a function  $u$  at a point  $\mathbf{x}$  is defined to be

$$\tilde{u}_{\mathbf{x}}(r) = \int_{\partial B(\mathbf{x}, r)} u \, dS = \int_{\mathbb{S}^{n-1}} u(\mathbf{x} + r\mathbf{y}) \, dS(\mathbf{y}),$$

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where  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$  is the unit sphere and the “barred” integral signs denote the average:

$$\bar{\int}_{\partial B(\mathbf{x}, r)} u \, dS = \frac{1}{A_n r^{n-1}} \int_{\partial B(\mathbf{x}, r)} u \, dS,$$

where  $A_n$  is the area of  $\mathbb{S}^{n-1}$ . Note that the second integral in the definition of  $\tilde{u}_{\mathbf{x}}(r)$  makes sense even for  $r < 0$ ; thus, we adopt this as the definition for all real  $r$  for which the integrand is defined on  $\mathbb{S}^{n-1}$ . We see that  $\tilde{u}_{\mathbf{x}}$  is an *even* function; it is  $C^k$  if  $u$  is  $C^k$ , and  $\tilde{u}_{\mathbf{x}}(0) = u(\mathbf{x})$ .

When  $\omega$  is a ball, we can rewrite (2) in terms of spherical averages: Noting that the volume of the ball  $B(\mathbf{x}, r)$  is  $A_n r^n/n$ , we find

$$\int_{B(\mathbf{x}, r)} \Delta u(\mathbf{y}) \, d^n \mathbf{y} = \frac{n}{r} \int_{B(\mathbf{x}, r)} \partial_{\nu} u(\mathbf{y}) \, dS(\mathbf{y}) = \frac{n}{r} \int_{\mathbb{S}^{n-1}} \partial_r u(\mathbf{x} + r\mathbf{y}) \, dS(\mathbf{y}),$$

where we can move the  $r$  derivative outside the integral, and arrive at

$$\int_{B(\mathbf{x}, r)} \Delta u(\mathbf{y}) \, d^n \mathbf{y} = \frac{n}{r} \tilde{u}'_{\mathbf{x}}(r). \quad (4)$$

Along with  $\tilde{u}_{\mathbf{x}}(0) = u(\mathbf{x})$ , this implies

**Theorem 2** (The mean value property of harmonic functions). *A  $C^2$  function  $u$  on a domain  $\Omega$  is harmonic if and only if  $\tilde{u}_{\mathbf{x}}(r) = u(\mathbf{x})$  for all  $\mathbf{x} \in \Omega$  and all  $r$  for which  $\bar{B}(\mathbf{x}, |r|) \subset \Omega$ .*

In general, we say a function  $u$  satisfies the *mean value property* if  $\tilde{u}_{\mathbf{x}}(r) = u(\mathbf{x})$  whenever  $\bar{B}(\mathbf{x}, |r|) \subset \Omega$ . We shall see below (Theorem 4) that the mean value property characterizes harmonic functions. But first, we collect an easy consequence of (4).

**Proposition 3.** *For any  $C^2$  function  $u$ , we have*

$$\Delta u(\mathbf{x}) = n \tilde{u}''_{\mathbf{x}}(0).$$

*Proof.* The function  $\tilde{u}_{\mathbf{x}}$  is even, so  $\tilde{u}'_{\mathbf{x}}(0) = 0$ . Therefore, letting  $r \rightarrow 0$  in (4), we arrive at the stated result. ■

**Theorem 4** (The mean value property and regularity). *Assume that a continuous function  $u$  satisfies the mean value property on a domain  $\Omega$ . Then  $u$  is infinitely differentiable, and is therefore harmonic. In particular, every harmonic function is infinitely differentiable.*

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*Proof.* This proof may seem long, but only because we use it to develop some tools that have wider applicability.

First, define a *standard mollifier*  $\rho: \mathbb{R}^n \rightarrow \mathbb{R}$ . Here is one of many possible definitions:

$$\rho(\mathbf{x}) = \begin{cases} ae^{1/(|\mathbf{x}|^2-1)}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \geq 1, \end{cases}$$

where  $a > 0$  is chosen to ensure that

$$\int_{\mathbb{R}^n} \rho \, d\mathbf{x} = 1.$$

That is one of the defining qualities of a standard mollifier. The others are: That  $\rho \geq 0$  everywhere, that it vanishes outside the unit ball, that it is infinitely differentiable, and is radially symmetric – that is, a function of  $|\mathbf{x}|$  alone.

For any  $\delta > 0$  we can squeeze the mollifier to fit inside a ball of radius  $\delta$ :

$$\rho_\delta(\mathbf{x}) = \frac{1}{\delta^n} \rho\left(\frac{\mathbf{x}}{\delta}\right),$$

so that  $\rho_\delta$  also has integral 1, but vanishes outside the ball  $B(0, \delta)$ .

Now we consider the convolution product

$$u * \rho_\delta(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{y}) \rho_\delta(\mathbf{x} - \mathbf{y}) \, d^n \mathbf{y}.$$

This is defined for all  $x \in \Omega$  with a distance less than  $\delta$  to the complement of  $\Omega$ . Thus, for any  $x \in \Omega$ , we can make  $\delta$  small enough so that  $u * \rho_\delta$  is defined at  $\mathbf{x}$ .

Moreover,  $u * \rho_\delta$  is infinitely differentiable: This is proved by differentiating with respect to the components of  $\mathbf{x}$  under the integral sign, as much as you like.

Finally, the mean value property of  $u$  and the radial symmetry of  $\rho_\delta$  combine to ensure that  $u(\mathbf{x}) = u * \rho(\mathbf{x})$  for all  $\mathbf{x}$  where  $u * \rho$  is defined, which is what we were going to prove.

For a detailed argument, write

$$u * \rho(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{x} - \mathbf{y}) \rho_\delta(\mathbf{y}) \, d^n \mathbf{y}$$

and write the integral in polar form:

$$\begin{aligned} u * \rho(\mathbf{x}) &= \int_0^\delta \int_{\partial B(\mathbf{x}, r)} u(\mathbf{x} - \mathbf{y}) \rho_\delta(\mathbf{y}) \, dS(\mathbf{y}) \, dr \\ &= \int_0^\delta \int_{\mathbb{S}^{n-1}} u(\mathbf{x} - r\mathbf{y}) \rho_\delta(r\mathbf{y}) \, dS(\mathbf{y}) \, r^{n-1} \, dr. \end{aligned}$$

Now use the radial symmetry:  $\rho_\delta(r\mathbf{y})$  is constant for  $\mathbf{y} \in \mathbb{S}^{n-1}$ , so this factor can be moved outside the inner integral. Next, use the mean value property of  $u$ :

$$\int_{\mathbb{S}^{n-1}} u(\mathbf{x} - r\mathbf{y}) \rho_\delta(r\mathbf{y}) \, d^n \mathbf{y} = A_n u(\mathbf{x}).$$

But  $u(\mathbf{x})$  is a constant, which we move outside the outer integral. We are left with

$$u * \rho(\mathbf{x}) = u(\mathbf{x}) \int_0^\delta r^{n-1} A_n \rho_\delta(r\mathbf{y}) \, dr$$

where  $\mathbf{y}$  is any unit vector. But running the whole calculation in reverse, this time without the  $u$  term, reveals that the integral here is merely the integral of  $\rho_\delta$ , which has the value 1. Thus we are left with  $u * \rho(\mathbf{x}) = u(\mathbf{x})$ , as claimed. ■

## The maximum principle

**Definition.** A  $C^2$  function  $u$  is called *subharmonic* if  $\Delta u \geq 0$ , and *superharmonic* if  $\Delta u \leq 0$ . Thus it is harmonic if and only if it is both subharmonic and superharmonic. (The reason for the naming will become clear later; see Corollary 7.) Clearly,  $u$  is superharmonic if and only if  $-u$  is subharmonic.

**Theorem 5** (Strong maximum principle). *Assume that  $u \in C^2(\Omega)$  is subharmonic in a region  $\Omega \subseteq \mathbb{R}^n$ . If  $u$  has a global maximum in  $\Omega$ , then  $u$  is constant.*

*Proof.* Let  $M$  be the global maximum of  $u$ , and put

$$S = \{\mathbf{x} \in \Omega \mid u(\mathbf{x}) = M\}.$$

Then  $S$  is a closed subset of  $\Omega$ , by the continuity of  $u$ . It is also nonempty by assumption.

Consider any  $\mathbf{x} \in S$ . From (4) and the subharmonicity of  $u$ , we get  $\tilde{u}'_x(r) \geq 0$  for  $r > 0$ . Thus we get  $\tilde{u}_x(r) \geq \tilde{u}_x(0) = u(\mathbf{x}) = M$  for  $r > 0$  (so long as  $\bar{B}(\mathbf{x}, r) \subset \Omega$ ). But  $u \leq M$  everywhere, and if  $u < M$  anywhere on the sphere  $\partial B(\mathbf{x}, r)$ , we would get  $\tilde{u}_x(r) < 0$ . Thus  $u$  is constant equal to  $M$  on  $\partial B(\mathbf{x}, r)$  for any small enough  $r$ , and therefore,  $u$  is constant in some neighbourhood of  $\mathbf{x}$ . This means that  $S$  is open.

Since  $S$  is an open, closed, and nonempty subset of the connected set  $\Omega$ , we must have  $S = \Omega$ , and the proof is complete. ■

**Remark.** Obviously, we obtain a strong *minimum* principle for *superharmonic* functions by multiplying by  $-1$ . In particular, a non-constant harmonic function cannot attain a minimum or maximum value anywhere in  $\Omega$ .

**Corollary 6** (Weak maximum principle). *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain, and assume that  $u \in C(\overline{\Omega}) \cap C^2(\Omega)$  is subharmonic. Then*

$$\max\{u(\mathbf{x}) \mid \mathbf{x} \in \overline{\Omega}\} = \max\{u(\mathbf{x}) \mid \mathbf{x} \in \partial\Omega\}.$$

*In particular, a harmonic function which is continuous on  $\overline{\Omega}$  attains its minimum and maximum values on the boundary  $\partial\Omega$ .*

*Proof.* The weak principle is an obvious consequence of the strong principle. However, it is worth noting that a much more elementary proof exists. Namely, for any  $\varepsilon > 0$ , let  $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$ , and note that then  $\Delta v > 0$ . But  $\Delta v(\mathbf{x}) \leq 0$  if  $\mathbf{x}$  is an interior minimum point for  $v$ , so  $v$  cannot have any maximum in the interior. Thus for any  $\mathbf{x} \in \Omega$ ,

$$u(\mathbf{x}) = v(\mathbf{x}) - \varepsilon|\mathbf{x}|^2 \leq \max_{\partial\Omega} v \leq \max_{\partial\Omega} u + \varepsilon \max_{\mathbf{x} \in \partial\Omega} |\mathbf{x}|^2.$$

Now let  $\varepsilon \rightarrow 0$  to arrive at the conclusion  $u(\mathbf{x}) \leq \max_{\partial\Omega} u$ . ■

Our next result explains the terms *sub-* and *superharmonic*: A subharmonic function is below, and a superharmonic above, a harmonic function given the same boundary data.

**Corollary 7.** *Assume that  $\Omega$  is a bounded domain, that  $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ , that  $u$  is harmonic in  $\Omega$ , and that  $v = u$  on  $\partial\Omega$ . If  $v$  is subharmonic, then  $v \leq u$  in  $\Omega$ , while if  $v$  is superharmonic, then  $v \geq u$  in  $\Omega$ .*

*Proof.* Apply the weak maximum principle to  $v - u$  if  $v$  is subharmonic, or to  $u - v$  if  $v$  is superharmonic. ■

**Remark.** Corollary 7 suggests a strategy for proving existence of a solution to the Dirichlet problem for the Laplace equation: Assume we are trying to find a harmonic function on  $\Omega$  with the given boundary value  $g$ . Consider the pointwise *supremum* of all subharmonic functions which are  $\leq g$  on  $\partial\Omega$ , and the pointwise *infimum* of all superharmonic functions which are  $\geq g$  on  $\partial\Omega$ . If the two functions coincide, they should provide a solution to the problem. This is the basis for *Perron's method*, which we will hopefully get a look at later.