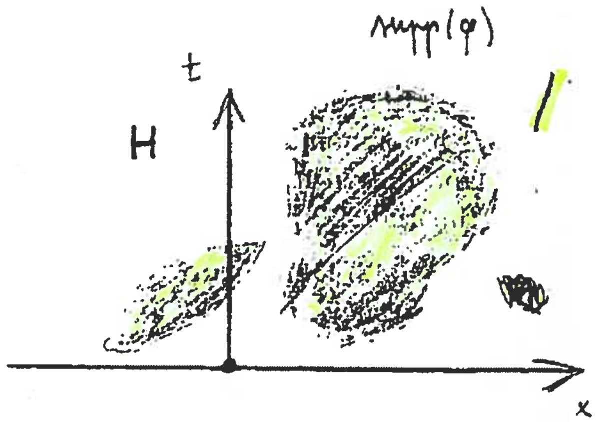
Weak solutions to Laws of conservation

Ren.-Rog. Chapter 3

Consider the equation

$$\boxed{u_t + \frac{\partial}{\partial x} f(u) = 0,}$$



or, more generally, the system

$$\bar{u}_t + \bar{f}(\bar{u})_x = \bar{0},$$

where $\bar{u} = (u_1, \dots, u_N)$, $\bar{f} = (f_1, \dots, f_N)$, $f_j = f_j(u_1, \dots, u_N)$, and $u_j = u_j(x,t)$, $j = 1, 2, \dots, N$. We are mainly interested in solutions defined in the <u>upper half-plane</u>

$$H = \{(x,t) \mid t > 0, \ -\infty < x < \infty\}.$$

We assume that all the first derivatives $D_k f_j$ are continuous in H.

TEST - FUNCTIONS

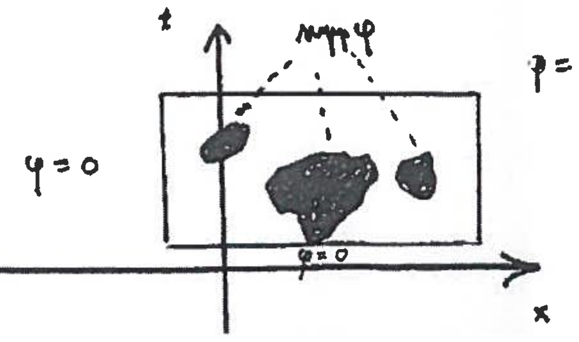
We say that $\varphi \in C_0^k(H)$, if $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ has continuous derivatives $D^\alpha \varphi$ of all orders $|\alpha| \leq k$ and if

The bar denotes the <u>closure</u> of the set.

$$\operatorname{supp}(\varphi) = \overline{\{(x,t) \mid \varphi(x,t) \neq 0\}} \subset H$$

In practical terms, φ vanishes identically outside some rectangle in the upper half-plane H (the rect. does not touch the x-axis). The functions $\varphi \in C_0^\infty(H)$ will be used as test-functions.

<u>Remark</u>: It does not matter wheather $\varphi(x,t)$ be defined for $t < 0$ or not.



DEF.: We say that $\bar{u}$ is a weak solution in H, if

$$\boxed{\int_0^\infty \int_{-\infty}^\infty [\bar{u}(x,t)\,\phi_t(x,t) + \bar{f}(\bar{u}(x,t))\,\phi_x(x,t)]\,dt\,dx = \bar{0}}$$

$t>$

whenever $\varphi \in C_0^\infty(H)$. (A priori, we assume that $u_j \in L^1_{loc}(\Omega)$, i.e., $\int_\varepsilon^T \int_a^b |u_j|\,dt\,dx < \infty$ for $0<\varepsilon<T<\infty$, $-\infty < a < b < \infty$, so that the above integral is well defined.)

– The set where $\phi = 0$ does not touch the x-axis!

Remarks 1°) The actual set of integration is only the support of φ, or, conveniently $\varepsilon \le t \le T$, $a \le x \le b$, the limits of integration depending on ϕ.

2°) One gets the same concept of weak solutions, if the test-functions φ are chosen from the class $C_0^1(H)$. (This class is wider.)

3°) DEFINITION The weak solution $\bar{u}$ is said to take the boundary values "$\bar{u}(x,0) = \bar{u}_0(x)$" in the weak sense, if

$$\int_0^\infty \int_{-\infty}^\infty [\bar{u}(x,t)\phi_t(x,t) + \bar{f}(\bar{u}(x,t))\,\phi_x(x,t)]\,dt\,dx$$

$$+ \int_{-\infty}^\infty \bar{u}_0(x)\,\phi(x,0)\,dx = \bar{0}$$

whenever $\varphi \in C_0^\infty(\mathbb{R}^2)$. As before, φ is identically zero outside a circle centered at the origin (or a rectangle), but $\varphi(x,0)$ may now differ from zero on a finite portion of the x-axis.

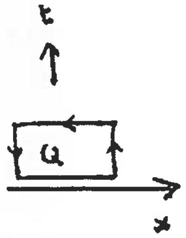
The point is that <u>classical solutions are weak solutions</u>, but that there are physically interesting "solutions" that are weak solutions (not classical! Some of them are discontinuous so that they do not even have the derivatives prescribed by the equation itself.)

Suppose that $\bar{u}$ has continuous first derivatives in H. If $\varphi \in C_0^\infty(H)$, then

$$(\bar{u}\phi)_t + (\bar{f}(\bar{u})\phi)_x = \underbrace{(\bar{u}_t + \bar{f}(\bar{u})_x)}_{\text{This is } \bar{0} \text{ for classical solutions!}}\phi + \bar{u}\phi_t + \bar{f}(\bar{u})\phi_x$$

LEMMA <u>Suppose that $\bar{u}$ is a classical solution in H, i.e., $\bar{u}$ has continuous first derivatives and $\bar{u}_t + \bar{f}(\bar{u})_x = \bar{0}$. Then u is a weak solution.</u>

<u>Proof</u>: Choose any $\varphi \in C_0^\infty(H)$ and fix an open rectangle $Q \subset H$ such that $\varphi \equiv 0$ in $H \setminus Q$. By the identity above



$$\int_0^\infty \int_{-\infty}^\infty [\bar{u}\phi_t + \bar{f}(\bar{u})\phi_x]\,dt\,dx = \iint_Q \left[\frac{\partial}{\partial x}(\bar{f}(\bar{u})\phi) + \frac{\partial}{\partial t}(\bar{u}\phi)\right]dt\,dx$$

$$= \oint_{\partial Q} (-\bar{u}\phi)\,dx + \bar{f}(\bar{u})\phi\,dt = \bar{0},$$ since φ is 0 on the boundary ∂Q of Q

Green's formula

This is the condition in the

Remark: If Q is chosen so that one side is on the x-axis and if $\phi \in C_0^\infty(\mathbb{R}^2)$, then the previous calculation will produce the identity

$$\int_0^\infty \int_{-\infty}^{\infty} [\bar{u}\phi_t + \bar{f}(\bar{u})\phi_x]\,dt\,dx = -\int_{-\infty}^{\infty} \bar{u}(x,0)\,\phi(x,0)\,dx$$

for any classical solution taking the boundary values $\bar{u}(x,0)$ in a continuous way. This identity is the motivation for $\underbrace{\text{boundary}}_{\text{the definition of}}$ values in the weak sense.

Ex.: If ψ is a continuous function of one variable, then $u = u(x,t) = \psi(x-ct)$ is a weak solution to the equation

$$u_x + \frac{1}{c} u_t = 0.$$

For example, $u(x,t) = |x-ct|$ is such a solution. It is not differentiable along the line $x = ct$.

Ex.: Burgers' eqn $\quad u_t + \left(\frac{u^2}{2}\right)_x = 0$. Define

$$u(x,t) = \begin{cases} 1, & x \le \frac{t}{2} \\ 0, & x > \frac{t}{2} \end{cases}$$

This IS a weak solution in the upper half plane. VERIFY THIS! The initial profile propagates to the right with speed $1/2$. OBS! If the line $2x = t$ is replaced by some other line (or curve), the corresponding formula will not define a solution, not even a weak one. Verify that $u(x,t) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$ is NOT a weak solution!

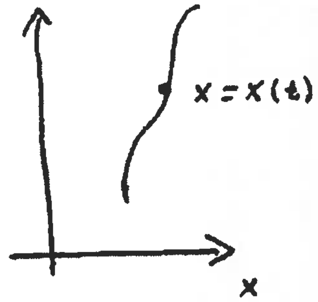
The equation $u_t + f'(u)u_x = 0$. Burgers' equation.

To integrate the equation

$$u_t + f'(u)\,u_x = 0$$

with initial conditions $u(x,0) = u_0(x)$ we observe that

$$\frac{d\,u(x(t),t)}{dt} = u_x \frac{dx}{dt} + u_t \cdot 1 \overset{?}{=} f'(u)\,u_x + u_t = 0$$



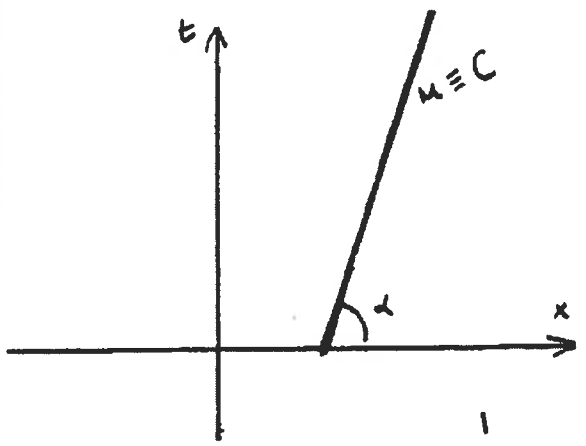
We obtain the conditions

$$\begin{cases} \dfrac{dx}{dt} = f'(u(x(t),t)) \\ u(x,0) = u_0(x), \quad u(x(t),t) = C(\text{onstant}) \end{cases}$$

The derivative with respect to t is zero by the construction!

Hence $\dfrac{dx}{dt} = f'(C)$ so that $\boxed{x = f'(C)\,t + x(0)}$

The characteristic curves are straight lines

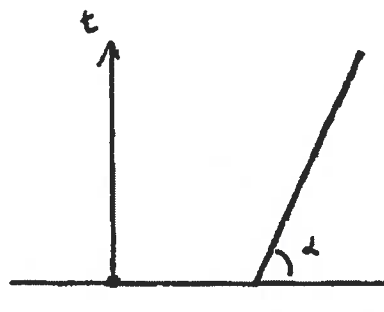


$$\tan\alpha = \frac{1}{f'(C)}$$

Given f, this is in principle

This is especially appealing for BURGERS' EQN

$$u_t + u\,u_x = 0$$

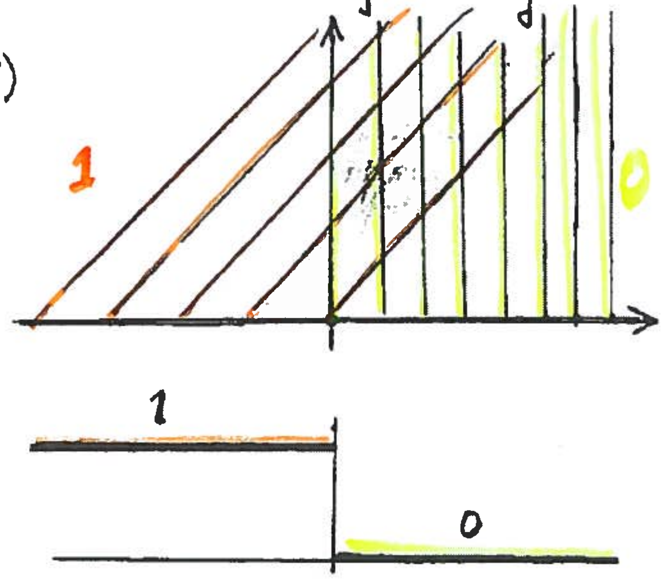


$u \equiv C$ on the line with slope C^{-1}.

$$\tan\alpha = C^{-1}$$

To see the limitation of classical solutions, consider the following situations for BURGER's eqn.

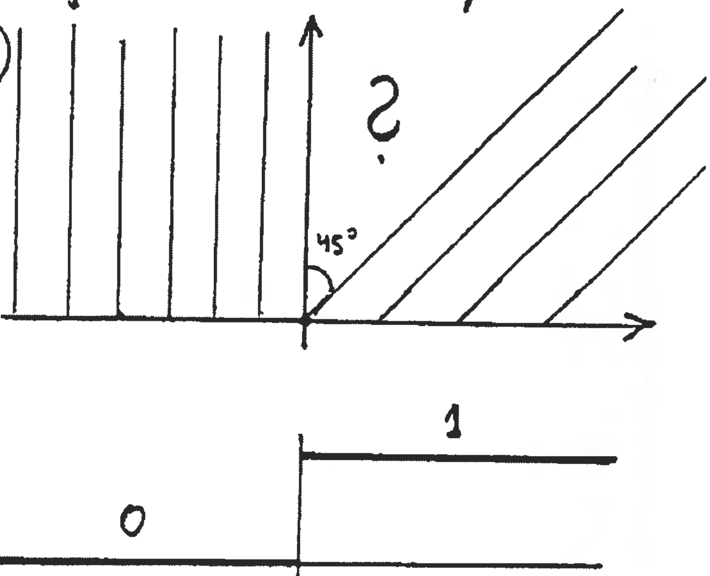
1°)



$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

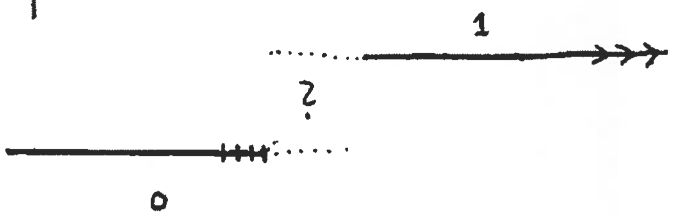
Intersecting characteristics! No classical solution can be obtained in the shadowed sector of 45°.

2°)

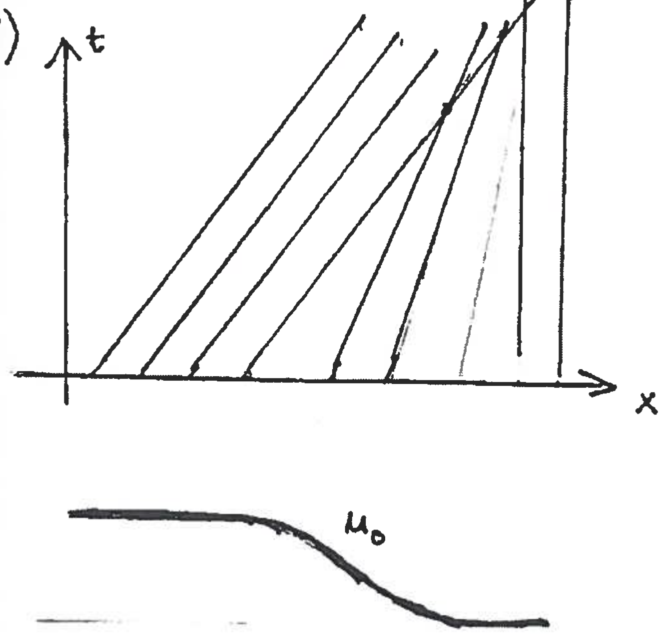


$$u(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

There is a dead sector of 45° that the characteristics cannot reach! After a while we have



3°)



Smooth initial values cause intersecting characteristics. There is no classical solution defined in the whole H.

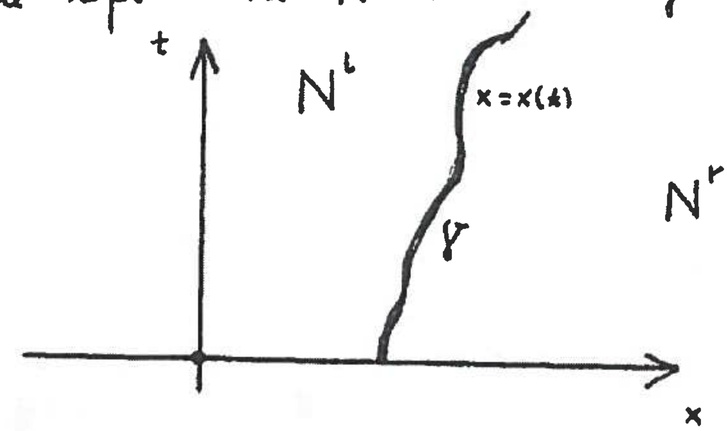
<u>Remark</u>: $u_t + f(u)_x = 0$

If f is convex ($f'' > 0$) and if the initial values $u_0(x)$ are not monot. increasing, then the characteristics will intersect. There is no classical solution in the upper half-plane (but it exists near the x-axis).

The Rankine-~~Hugionot~~ Condition

- HUGONIOT

Let $\bar{u}$ be a weak solution to the eqn $\boxed{\bar{u}_t + \bar{f}(\bar{u})_x = \bar{0}}$ in the upper half-plane. Suppose that the curve $x = x(t)$ divides H in two domains: N^l to the left and N^r to the right of the curve.



γ is the curve with eqn. $x = x(t)$

We assume that

$$\boxed{s = \frac{dx}{dt}}$$

is continuous.

Suppose now that

1°) $\bar{u}$ is a classical solution in N^l with continuous boundary values on the curve

$\bar{u}$ is a classical solution in N^r with continuous boundary values on the curve

NOTE The left and right boundary values are not necessarily the same! There is a jump (or a shock)

2°) The jump $[\bar{u}] = \lim_{N^r} \bar{u} - \lim_{N^l} \bar{u}$ is continuous along the curve. Here

$$[\bar{u}] = \lim_{\substack{(x,t)\to(x(t_0),t_0)\\(x,t)\in N^r}} \bar{u}(x,t) - \lim_{\substack{(x,t)\to(x(t_0),t_0)\\(x,t)\in N^l}} \bar{u}(x,t)$$

is the difference between the right and left boundary

3°) Recall that $\bar{u}$ is a weak solution in the whole upper plane. (This will enforce a strong interaction between the two separate pieces.)

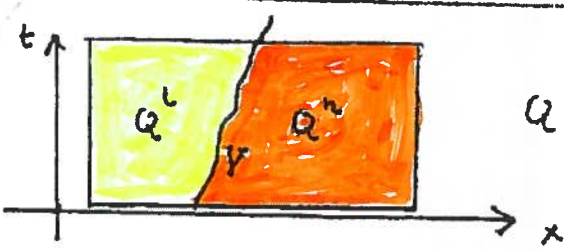
$$\boxed{s = \frac{dx}{dt}}$$

THM If all this is fulfilled, then

$$\boxed{[\bar{u}]\,\frac{dx}{dt} = [\bar{f}(\bar{u})]}$$

RANKINE – HUGONIOT

along the curve.



Ren–Rog. pp. 81–82.

Proof: Fix $\varphi \in C_0^\infty(H)$. Then $\operatorname{supp}(\varphi) \subset Q$

$$\bar{0} = \int_0^{\infty}\int_{\infty}^{-\infty}[\bar{u}\,\phi_t + \bar{f}(\bar{u})\,\phi_x]\,dt\,dx = \iint_{N^r} -\|- + \iint_{N^l} -\|-$$

$$= \iint_{N^r}\left[\frac{\partial}{\partial x}(\bar{f}(\bar{u})\phi) + \frac{\partial}{\partial t}(\bar{u}\phi)\right]dt\,dx + \iint_{N^l} -\|-$$

GREEN'S IDENT.

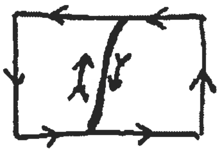
$$= \oint_{\partial Q^r} \bar{f}(\bar{u})\phi\,dt - \bar{u}\phi\,dx + \oint_{\partial Q^l} \bar{f}(\bar{u})\phi\,dt - \bar{u}\phi\,dx$$

$\phi = 0$ except on the curve

$$= \int_{\gamma,\,\infty}^{0} \bar{f}(\bar{u})\phi\,dt - \bar{u}\phi\,dx + \int_{\gamma,\,0}^{\infty} -\|-$$

$$= -\int^{\infty} [f(u)]\,\phi\,dt - [u]\,\phi\,dx$$

THE CURVE γ WILL BE TRAVERSED IN OPPOSITE DIRECTIONS!



Using the equation $x = x(t)$ describing the curve we arrive at

[The point is that the $\int$ is 0 for all test-functions!]

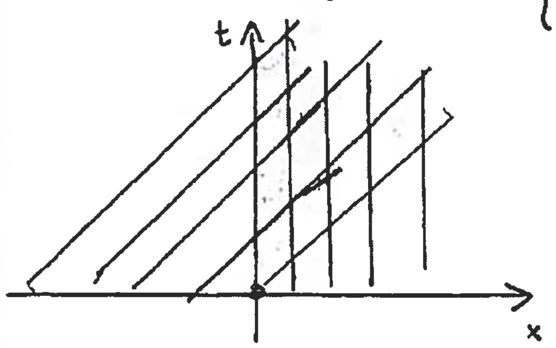
$$-\bar{0} = \int_0^\infty \underbrace{\left\{ [\bar{f}(\bar{u})] - [\bar{u}] \frac{dx}{dt} \right\}}_{\text{This is a continuous function of } t} \phi(x(t), t)\, dt \qquad [\ ] = \text{right} - \text{left}$$

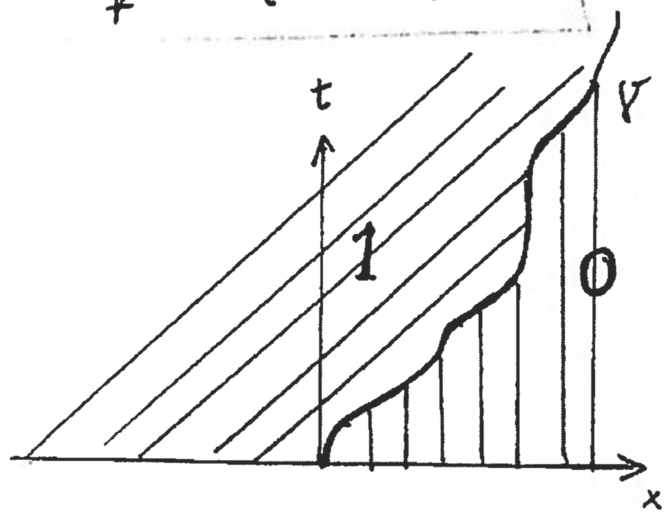
†) See page 24.

By the VARIATIONAL LEMMA†), the expression $\{\ \} = 0$. This is the Rankine-Hugoniot condition. □

<u>Example</u> Consider $\boxed{\text{Burgers' eqn} \quad u_t + u u_x = 0}$ with the initial data

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$$





$\gamma: \quad x = x(t)$

How should γ be drawn?

$$s = \frac{dx}{dt}$$

$$s(1-0) = \frac{1^2}{2} - \frac{0^2}{2} \qquad \text{Rankine-Hugoniot}$$

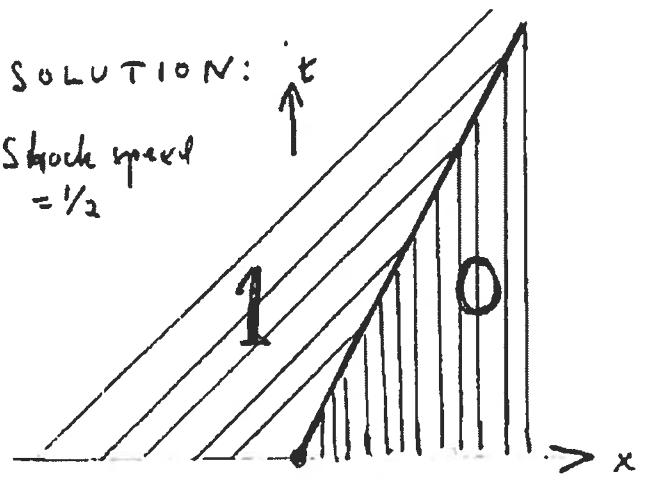
$$s = \frac{1}{2}$$

$$\frac{dx}{dt} = \frac{1}{2}, \qquad 2x = t + \text{Const.}$$

γ is a straight line!

SOLUTION:

Shock speed $= 1/2$

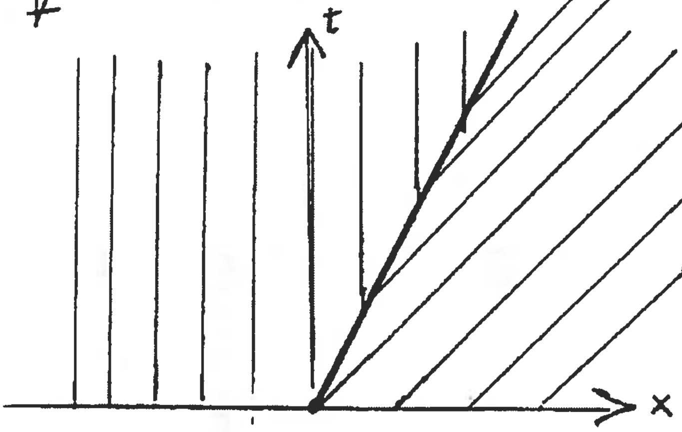


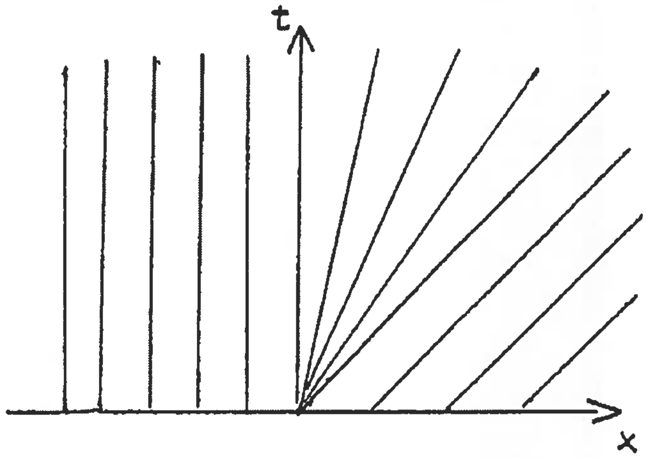
centered expansion wave, fan, release wave, rarefaction wave

Unfortunately, the same initial values can give rise to several different weak solutions. For example, the initial values

$$u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

yield the following two solutions to Burgers' equation $u_t + uu_x = 0 \quad (t > 0,\ -\infty < x < \infty)$





The <u>shock</u> <u>solution</u>

$$u(x,t) = \begin{cases} 0, & x < \frac{t}{2} \\ 1, & x \geq \frac{t}{2} \end{cases}$$

obtained from the Rankine-Hugoniot condition.
(RULED OUT BY LAX' CONDITION)

The <u>rarefaction</u> <u>wave</u>
(the <u>fan</u> <u>wave</u>)

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x < t \\ 1, & x \geq t \end{cases}$$

This solution is continuous.
Verify that this is a <u>weak</u> solution

Indeed, there are many more solutions.

From this multiplicity of solutions "the right one" must be picked on the basis of physical considerations. Several mathematical devises have been proposed, as, for example, the <u>Lax Shock Condition</u>, <u>Entropy Conditions</u>, and <u>Viscosity</u> <u>Solutions</u>.

Read § 3.5.1 in Ren. & Rog. about Riemann problems for single equations

<u>Lax Shock Condition</u> Suppose that the weak solution u to the single conservation law $u_t + f'(u)u_x = 0$ has a shock along a curve γ: $x = x(t)$ obtained from the Rankine-Hugoniot Shock Condition. Then we require that

LAX: $$\boxed{f'(u^l) > s > f'(u^r), \quad s = \frac{dx}{dt}}$$

along the curve γ. For Burgers' equation this means that $u^l > s > u^r$, so that one can only "jump down" along a shock.

Burger

In the previous example the shock solution is ruled out by the Lax condition but the fan wave is admitted!

For systems $\bar{u}_t + \bar{f}(\bar{u})_x = 0$, the formulation of the Lax Shock Condition is quite complicated.

<u>The Entropy Condition</u> Additional conservation laws are used to rule out "impossible" solutions.

<u>Viscosity Solutions</u> Burgers' equation $u_t + uu_x = 0$ is a degenerated form of the equation

$$u_t + uu_x = \varepsilon u_{xx},$$

obtained as the constant $\varepsilon \longrightarrow 0$. The complete eqn has unique solutions with given initial values, if some reasonable conditions are prescribed. The viscosity solution is the limit (in some weak sense) of these solutions u^ε as $\varepsilon \longrightarrow 0$.

Peter Lindqvist 1993

The Variational Lemma

The auxiliary result below is fundamental for the theory of (elliptic) partial differential equations. It was known to Euler and Lagrange. (An "integrated version" is du Bois-Reymond's lemma.)

THE VARIATIONAL LEMMA <u>Suppose</u> <u>that</u> $g : \Omega \longrightarrow \mathbb{R}$ <u>is</u> <u>a</u> <u>continuous</u> <u>function</u> <u>defined</u> <u>in</u> <u>an</u> <u>open</u> <u>set</u> $\Omega \subset \mathbb{R}^n$. <u>If</u>

$$\int_\Omega g(x)\eta(x)\,dx = 0$$

<u>whenever</u> $\eta \in C_0^\infty(\Omega)$, <u>then</u> $g \equiv 0$. (The notation means that all derivatives $D^\alpha\eta$ exist and that the closure of the set where $\eta \neq 0$ is a compact subset of Ω).

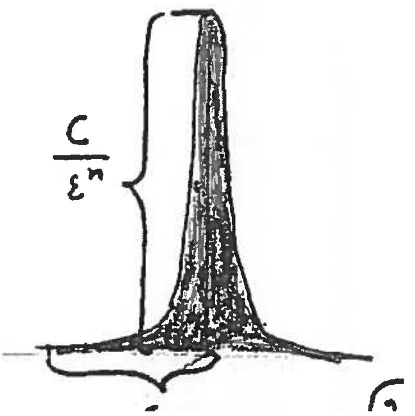
PROOF: There are elementary proofs, but we shall use the following device. Define <u>Friedrich's</u> <u>mollifier</u>

$$\varrho_\varepsilon(x) = \begin{cases} \dfrac{C}{\varepsilon^n} e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & \text{when } |x| < \varepsilon \\ 0, & \text{when } |x| \geq \varepsilon. \end{cases}$$

ϱ_ε concentrates the total mass 1 near the origin!

The constant C is chosen so that

$$C = \int_{|x|<1} e^{-\frac{1}{1-|x|^2}}\,dx.$$



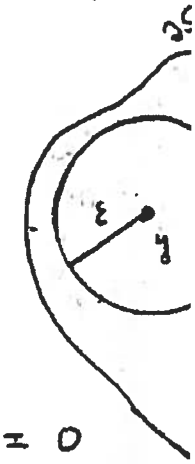
Then $\int \varrho_\varepsilon(x)\,dx = 1$ (Calculate!).

$x = (x_1, \dots, x_n)$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ $\int \dots dx = \int \dots \int \dots dx_1 \dots dx_n$.

Fix any $y \in \Omega$. When $0 < \varepsilon < \operatorname{dist}(y, \partial\Omega)$,

$$\eta_\varepsilon = \eta_\varepsilon(x) = \varrho_\varepsilon(x-y)$$

will do as test-function. By the assumption



$$\int g(x)\varrho_\varepsilon(x-y)\,dx = 0.$$

We claim that $g(y) = \lim\limits_{\varepsilon \to 0+} \int g(x)\varrho_\varepsilon(x-y)\,dx$ ($= 0$ by the assumption). Indeed

$$\left| g(y) - \int g(x)\varrho_\varepsilon(x-y)\,dx \right| = \left| \int (g(y)-g(x))\varrho_\varepsilon(x-y)\,dx \right|$$

$$\leq \int\limits_{|x-y|\leq\varepsilon} |g(y)-g(x)|\varrho_\varepsilon(x-y)\,dx \leq \max_{\substack{x \\ |x-y|\leq\varepsilon}} |g(y)-g(x)| \cdot \underbrace{\int \varrho_\varepsilon(x-y)\,dx}_{=1}$$

$$= \max_{\substack{x \\ |x-y|\leq\varepsilon}} |g(y)-g(x)| \xrightarrow[\varepsilon \to 0+]{} 0,$$ since

$\lim\limits_{x \to y} g(x) = g(y)$ by the continuity of g. $\square$

<u>Remark</u> If we merely assume that $g \in L^1_{loc}(\Omega)$, i.e. that

$$\int_K |g(x)|\,dx < \infty$$

whenever $K \subset\subset \Omega$, then $g = 0$ almost everywhere in Ω. This means that $\int_\Omega |g(x)|\,dx = 0$.