TMA 4305 19. $\overline{\text{XII}}$. 2016 Peter Lindqvist

PARTIAL DIFFERENTIAL EQUATIONS.

① By Duhamel's Principle the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + t x^2 & (-\infty < x < \infty,\ t > 0) \\ u(x,0) = 0 \end{cases}$$

is given by

$$u(x,t) = \int_0^t \frac{ds}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4(t-s)}} s y^2 \, dy$$

$$= \cdots\cdots = \frac{t^3}{3} + \frac{x^2 t^2}{2}.$$

The answer can also be found by a "clever Ansatz."

② The parabolic maximum principle states that the maximum of the solution is attained on the parabolic boundary Γ_T of Ω_T

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$$

where $\Gamma_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T]$.

For the proof, consider

$$v(x,t) = u(x,t) - \frac{\varepsilon}{T-t}, \qquad \varepsilon > 0.$$

Now

$$v_t = v_{xx} + 2v_{yy} - \frac{\varepsilon}{(T-t)}$$
$$\le v_{xx} + 2v_{yy} - \frac{\varepsilon}{T} < v_{xx} + 2v_{yy}.$$

At an interior maximum of v, we must have $v_t = 0$, $v_{xx} \le 0$, $v_{yy} \le 0$, which contradicts $v_t < v_{xx} + 2v_{yy}$. Hence v has no interior maximum. On the other hand,

$$v(x, T-) = -\infty.$$

Thus a maximum is out of the question when $t = T$.

It follows that

$$u(x,t) \le \max_{\bar{\Omega}_T} v + \frac{\varepsilon}{T-t}$$
$$= \max_{\Gamma_T} v + \frac{\varepsilon}{T-t}$$
$$\le \max_{\Gamma_T} u + \frac{\varepsilon}{T-t}$$

Send $\varepsilon \to 0$. Then $u(x,t) \le \max_{\Gamma_T} u$. $\square$

Remark: Also $v = u - \varepsilon t$ will do, but then the values $v(x,T)$ require to be addressed.

(3) For the difference $w = u_2 - u_1$ of two solutions we have

$$\begin{cases} w_{tt} - c^2 \Delta w + 3w = 0 \\ w(x,0) = 0, \quad w_t(x,0) = 0 \end{cases}$$

We get

$$\frac{d\mathcal{E}(t)}{dt} = \iiint (w_t w_{tt} + c^2 \nabla w \cdot \nabla w_t + 3 w w_t)\, dx\,dy\,dz$$

Now

$$\iiint \nabla w \cdot \nabla w_t \, dx\,dy\,dz = \oiint w_t \frac{\partial w}{\partial n}\, dS$$

$$- \iiint w_t \Delta w \, dx\,dy\,dz$$

where we integrate over a ball with a radius so large that the surface term vanishes. Thus

$$\frac{d\mathcal{E}(t)}{dt} = \iiint w_t (\underbrace{c^2 \Delta w - 3w - c^2 \Delta w + 3w}_{\equiv 0})\, dx\,dy\,dz$$

$$= 0.$$

Hence $\mathcal{E}(t) = \mathcal{E}(0)$, a constant. Since $w(x,0) \equiv 0$, $\nabla w(x,0) = 0$. We see that $\mathcal{E}(0) = 0$. So is $\mathcal{E}(t)$. It follows that $w = 0$, since all the terms in $\mathcal{E}(t)$ are $\geqq 0$.

$$w = u_2 - u_1 = 0 \implies u_2 = u_1 \qquad \square$$

(4) Start from

$$\int_{-\infty}^{\infty} u(x)\left[\phi''(x) - k^2\phi(x)\right] dx = -\phi(0)$$

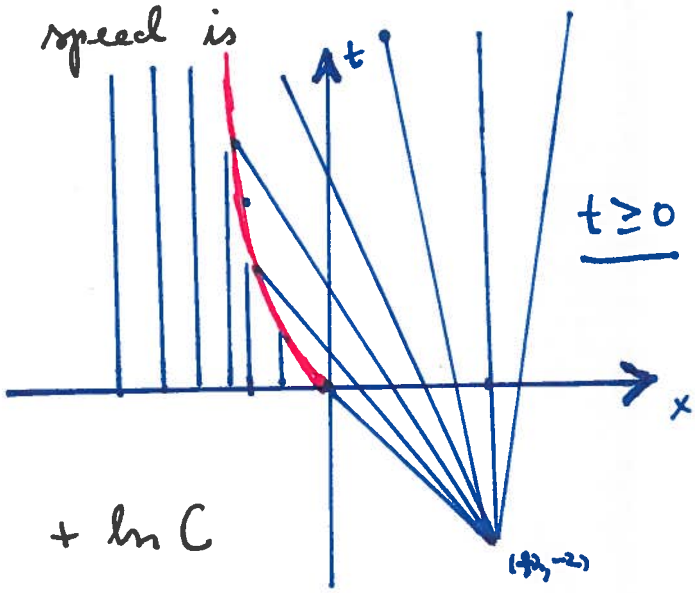
and integrate by parts in

$$\int_{-\infty}^{\infty} u(x)\,\phi''(x)\,dx = C\int_{-\infty}^{0} e^{-\alpha x}\phi''(x)\,dx + C\int_{0}^{\infty} e^{\alpha x}\phi''(x)\,dx$$

to achieve

$$u(x) = -\frac{1}{2k}e^{k|x|} \quad \text{or} \quad +\frac{1}{2k}e^{-k|x|}.$$

(5) The Rankine – Hugoniot shock condition requires that the shock speed is



$$\frac{dx}{dt} = \frac{1}{2}\,\frac{x-2}{t+2}.$$

Then

$$\frac{2\,dx}{x-2} = \frac{dt}{t+2}$$

$$\ln(x-2)^2 = \ln|t+2| + \ln C$$

$$(x-2)^2 = 2(t+2), \quad x \le 0, \quad t \ge 0$$

The shock is the parabola

$$x = 2 - \sqrt{2t+4}, \quad t \ge 0$$

⑥ Due to a misprint example 6a is impossible. In the favour of the student, 6a does not count at all in the total evaluation.

(6b) Let $\eta \in C_0^\infty(Q)$ and consider the competing function

$$v(x) = u(x) + \varepsilon\eta(x)$$

The necessary condition

$$\left[\frac{d}{d\varepsilon} I(u+\varepsilon\eta)\right]_{\varepsilon=0} = 0$$

takes the form

$$\int_0^1\int_0^1 \left(2e^{2x}u_x\eta_x + u_x\eta_y + u_y\eta_x + 2u_y\eta_y - 6u^2\eta\right)dx\,dy = 0$$

and so (integration by parts)

$$-2\int_0^1\int_0^1 \eta\left(\frac{\partial}{\partial x}e^{2x}u_x + u_{xy} + u_{yy} + 3u^2\right)dx\,dy = 0$$

valid for all test functions η

from which we conclude that

$$\boxed{e^{2x}u_{xx} + u_{xy} + u_{yy} + 2e^{2x}u_x + 3u^2 = 0}$$

This is the Euler-Lagrange equation.

We claim that $u \geq 0$. If not, there exists an interior minimum point at which

$u < 0, \quad u_x = 0, \quad u_y = 0, \quad u_{xx} \geq 0,$

$$u_{yy} \geq 0 \quad \text{and} \quad \underbrace{u_{xx} u_{yy} - u_{xy}^2 \geq 0}_{\Rightarrow |u_{xy}| \leq \frac{u_{xx} + u_{yy}}{2}}$$

At the minimum point

$$e^{2x} u_{xx} + u_{xy} + u_{yy} + \overbrace{2e^{2x} u_x}^{=0} + 3u^2$$

$$\geq u_{xx} + u_{xy} + u_{yy} + 3u^2$$

$$\geq \frac{u_{xx} + u_{yy}}{2} + 3u^2 \geq 3u^2 > 0,$$

which contradicts the Euler–Lagrange equation.

Remark The misprint was $-2u^{\cancel{3}})\,dxdy$. In fact, the "minimum" of the integral is now $-\infty$.