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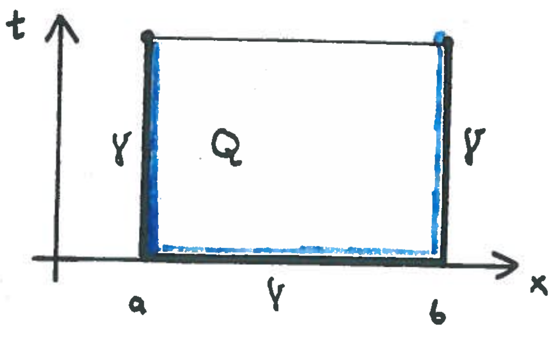
The simplest case of

Maximum Principles for Parabolic Equations

If $u_{xx} - u_t > 0$ in a domain in the xt-plane, then u cannot have an interior maximum point. (Because $u_t = 0$ and $u_{xx} \leq 0$ at such a point. This contradicts the differential inequality.) Roughly speaking, this phenomenon is the cause of the parabolic maximum principles.

Let Q denote the open rectangle $a < x < b$ and $0 < t < T$. Its closure $\overline{Q}$ is the closed set $a \leq x \leq b$, $0 \leq t \leq T$. The parabolic boundary γ consists of three closed segments: $\{(x,0) \mid a \leq x \leq b\}$, $\{(a,t) \mid 0 \leq t \leq T\}$, and $\{(b,t) \mid 0 \leq t \leq T\}$



Suppose that
$u \in C(\overline{Q})$, $u \in C^2(Q)$,
$u_{xx} - u_t \geqq 0$ in Q.
If $u \leq 0$ on the parabolic boundary γ, then $u \leq 0$ in $\overline{Q}$.

WEAK MAX. PRC.

Proof: If this weak maximum principle is true for every smaller rectangle $[a,b] \times [0,T']$, $T' < T$, then the result follows for $\overline{Q}$ by the continuity of u. Hence we may assume that the derivatives u_{xx} and u_t exist also at the latest time $t = T$, $a < x < b$. Define

$$v(x,t) = u(x,t) - \varepsilon t, \qquad \varepsilon > 0$$

Then

$$v_{xx} - v_t = u_{xx} - u_t + \varepsilon \geq \varepsilon > 0$$

so that v cannot have an interior maximum point. Let $\max\limits_{\overline{Q}} v = v(x_0, t_0)$. If $t_0 = T$ and $a < x_0 < b$, then $v_{xx} \leq 0$ and $v_t \geq 0$ at this point. (Why?). But this is a contradiction to the inequality $v_{xx} - v_t \geq \varepsilon$. Hence $(x_0, t_0) \in \gamma$.

Thus $u(x_0, t_0) \leq 0$ by the original assumption. Now

$$\begin{aligned} u(x,t) &= v(x,t) + \varepsilon t \\ &\leq v(x_0,t_0) + \varepsilon t \\ &= u(x_0,t_0) + \varepsilon(t - t_0) \\ &\leq 0 + \varepsilon(t - t_0) \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $u(x,t) \leq 0$ Q.E.D.

COMPARISON PRINCIPLE, UNIQUENESS

Cor. (Uniqueness) Suppose that u_1 and u_2 belong to $C(\overline{Q}) \cap C^2(Q)$ and that ~~u_1 solves~~ they are solutions to the heat equation $u_{xx} - u_t = 0$ in Q. If $u_2 \geq u_1$ on the parabolic boundary γ, then $u_2 \geq u_1$ in $\overline{Q}$.

If $u_2 = u_1$ on γ, then $u_2 = u_1$ in Q.

Proof: Consider the difference $u_2 - u_1$. $\square$

Remarks: 1°) If $m \le u(x,t) \le M$, when $(x,t) \in \gamma$, then $m \le u(x,t) \le M$ in $\bar{Q}$.

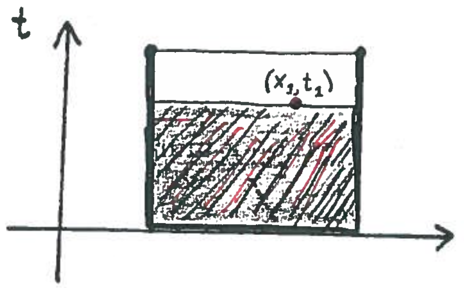
2°) If $|u_2(x,t) - u_1(x,t)| \le \varepsilon$, when $(x,t) \in \gamma$, then $|u_2(x,t) - u_1(x,t)| \le \varepsilon$ in $\bar{Q}$. This expresses stability with respect to boundary values on γ.

"subsolution"

"subcaloric"

"STRONG" MAXIMUM PRINCIPLE

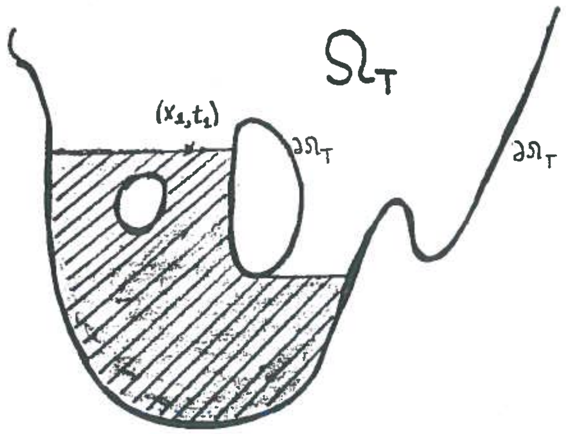
Theorem: Suppose that $u_{xx} - u_t \ge 0$ in Q and that $u \in C(\bar{Q}) \cap C^2(Q)$. Let $M = \max_{\bar{Q}} u$. If $M = u(x_1, t_1)$ at some interior point (x_1,t_1) $a < x_1 < b$ and $0 < t_1 < T$ or at some point $a < x_1 < b$ and $t_1 = T$, then $u(x,t) = M$, when $a \le x \le b$ and $0 \le t \le t_1$. (In other words, u attains its maximum at all "earlier points".)



If $u(x_1,t_1) = \mathrm{Max}\, u$, then $u \equiv M$ in the shadowed rectangle.

Proof: See Murray Protter & Hans Weinberger, Maximum Principles in Differential Eqs, 1967, 1984.

For more general domains in the (x,t)-plane the corresponding picture looks as follows:



This domain has 3 boundary components

Principle $\leftleftarrows\ \rightrightarrows$

$\downarrow$ (NOT $\uparrow$)