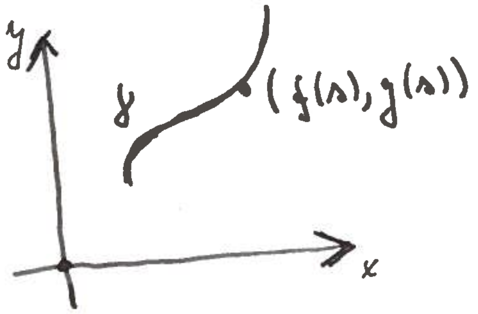
Classification of Second Order Equations

[Fritz John: "Partial Differential Equations", § 2.1. – Springer Verlag]

(not 2.6)

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$$



The Cauchy Data are given on the curve γ in the xy-plane

$$\begin{cases} u\big|_\gamma = h(s) \\ \dfrac{\partial u}{\partial x}\Big|_\gamma = \phi(s) \\ \dfrac{\partial u}{\partial y}\Big|_\gamma = \psi(s) \end{cases}$$

Cauchy-Data

(All calculations valid (only) along the curve γ.)

compatibility condition.

$$h(s) = u(f(s), g(s))$$

$$h'(s) = u_x f'(s) + u_y g'(s) = \phi(s) f'(s) + \psi(s) g'(s)$$

Try to determine u_{xx}, u_{xy}, u_{yy} along γ. Differentiate with respect to the parameter s.

$$\phi(s) = u_x(f(s), g(s)),$$

$$\phi' = u_{xx} f' + u_{xy} g'.$$

$$\psi' = u_{yx} f' + u_{yy} g'$$

$$\begin{cases} f' u_{xx} + g' u_{xy} = \phi' \\ f' u_{xy} + g' u_{yy} = \psi' \\ a u_{xx} + b u_{xy} + c u_{yy} = d \end{cases}$$

A linear system for the unknown quantities u_{xx}, u_{xy}, u_{yy} along γ.

This has a unique solution, if

$$D = \begin{vmatrix} f' & g' & 0 \\ 0 & f' & g' \\ a & b & c \end{vmatrix} = a g'^2 - b f' g' + c f'^2 \neq 0$$

in which case the curve γ is called non-characteristic

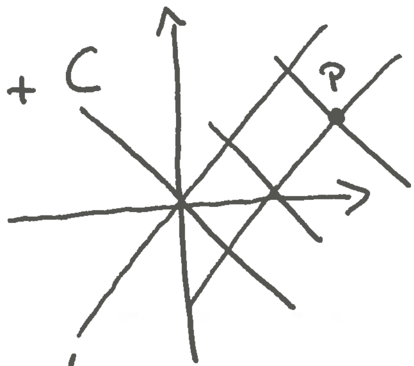
The equation for the characteristic curves is

$$\boxed{\begin{aligned} & a(dy)^2 - b\,dx\,dy + c(dx)^2 = 0 \\ & \frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$

EQN FOR THE CHARACTERISTIC CURVES

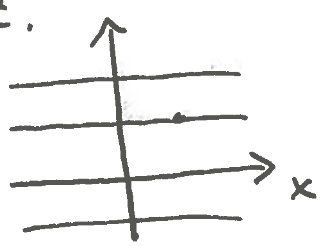
(1°) $4ac - b^2 < 0$ HYPERBOLIC There are two characteristics through each point

$u_{xx} - u_{yy} = 0$, $\frac{dy}{dx} = \pm 1$, $y = \pm x + C$



(2°) $4ac = b$ PARABOLIC There is one characteristic through each point.

$u_{xx} = u_y$, $\frac{dy}{dx} = 0$, $y = C$



(3°) $4ac - b^2 > 0$ ELLIPTIC No real characteristics

$u_{xx} + u_{yy} = 0$