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SCHRÖDINGER EQN.

$$\begin{cases} i u_t = u_{xx} & (-\infty < x < \infty) \\ u(x,0) = f(x) \end{cases}$$

(misprint in the book)

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx \qquad (t = \text{dumb})$$

$$\widehat{u_{xx}(x,t)} = (i\omega)^2\, \widehat{u(x,t)}, \quad \hat{u}(\omega,0) = \hat{f}(\omega).$$

Transform the Schrödinger equation:

$$-i \frac{\partial \hat{u}}{\partial t} = -\omega^2 \hat{u}$$

$$\hat{u}(\omega,t) = A(\omega)\, e^{-i\omega^2 t}, \quad A(\omega) = \hat{f}(\omega)$$

$$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-i\omega^2 t}$$

Auxiliary formula:

$$\frac{1}{\sqrt{2t}}\, e^{-i\frac{\pi}{4}}\, \widehat{e^{i\frac{x^2}{4t}}} = e^{-i\omega^2 t}$$

(see page 2)

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2t}}\, e^{-i\frac{\pi}{4}}\, \widehat{f(x)}\; \widehat{e^{i\frac{x^2}{4t}}}$$

$$= \frac{1}{\sqrt{2t}}\, e^{-i\frac{\pi}{4}}\, \frac{1}{\sqrt{2\pi}}\, \widehat{f(x) * e^{i\frac{x^2}{4t}}}$$

Taking the inverse (remove the "hats") we finally obtain the solution.

$$u(x,t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{i\frac{(y-x)^2}{4t}}\, dy$$

A wildly oscillating integral as $t \to \infty$.

<u>Remark</u> $-iu_t = u_{xx}$ is <u>formally</u> the Heat equation $u_t = k u_{xx}$, where $k = i$.

The formula

$$\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(y-x)^2}{4tk}}\, dy$$

yields formally for $k = +i$

$$\frac{1}{\sqrt{i}\,\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{+i\frac{(y-x)^2}{4t}}\, dy \qquad \left(-\frac{1}{i} = i\right)$$

Now $\sqrt{i} = \pm \frac{1+i}{\sqrt{2}} = e^{\pm i\frac{\pi}{4}}$. Two roots!

Only one of the roots can yield the right initial condition $\lim_{t\to 0+} u(x,t) = f(x)$. A special device is needed to discriminate!

$$\int_{-\infty}^{\infty} e^{+i\frac{x^2}{2}} e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} e^{i\left(\frac{x^2}{2} - \omega x + \frac{\omega^2}{2}\right)} e^{-i\omega^2/2}\, dx = e^{-i\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{i\left(\frac{x}{\sqrt{2}} - \frac{\omega}{\sqrt{2}}\right)^2} dx$$

$$= \sqrt{2}\, e^{-i\omega^2/2} \int_{-\infty}^{\infty} e^{iy^2} dy = \sqrt{2}\, e^{-i\omega^2/2} \sqrt{\pi}\, e^{+i\pi/4} \quad \text{(Fresnel's integral)}$$

Replace ω by $\sqrt{2t}\cdot\omega$. Recall

$$\widehat{f * g} = \sqrt{2\pi}\, \hat{f}\, \hat{g}$$