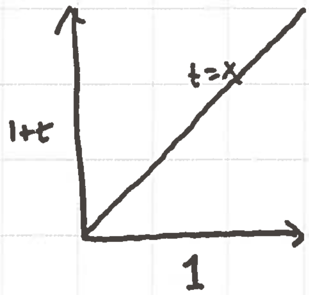
$$\begin{cases} u_t + u u_x = 0, & x > 0,\ t > 0 \\ u(x,0) = 1, & x > 0 \\ u(0,t) = 1 + t, & t > 0 \end{cases}$$



$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$ BURGERS' inviscid eqn

The characteristics emerging from the time axis are*)

$$t = s + \frac{x}{1+s}, \quad u = 1+s \qquad (s>0)$$

They have the envelope**) $x = \left(\frac{t+1}{2}\right)^2$, $t \geq 1$, which, as we shall see, is "below" the true shock curve. To find the shock, use

$$\begin{cases} \frac{dx}{dt} = \frac{(s+1)+1}{2} = 1 + \frac{s}{2} & \text{(Rankine–Hugoniot)} \\ x = (s+1)(t-s) \end{cases}$$

$$\begin{cases} dx = \left(1+\frac{s}{2}\right) dt \\ dx = (s+1)(dt - ds) + (t-s)\,ds \end{cases}$$

$$\left(1+\frac{s}{2}\right) dt = (s+1)\,dt + (t-2s-1)\,ds$$

$$\frac{s}{2}\,dt = (2s+1-t)\,ds$$

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*) Set up the characteristic equations, please.

**) The characteristics do not intersect in the sector $t > x$. Verify, please.

$$\frac{s}{2}\frac{dt}{ds} + t = 2s+1 \qquad (t = t(s))$$

This differential equation has the general solution

$$t = \frac{4}{3}s + 1 + C s^{-2}$$

We take $C = 0$ (otherwise $s \to 0+$ yields an impossibility.)

$$s = \frac{3}{4}(t-1)$$

$$x = (t-s)(s+1) = \frac{t+3}{4} \cdot \frac{3t+1}{4}$$

This yields the

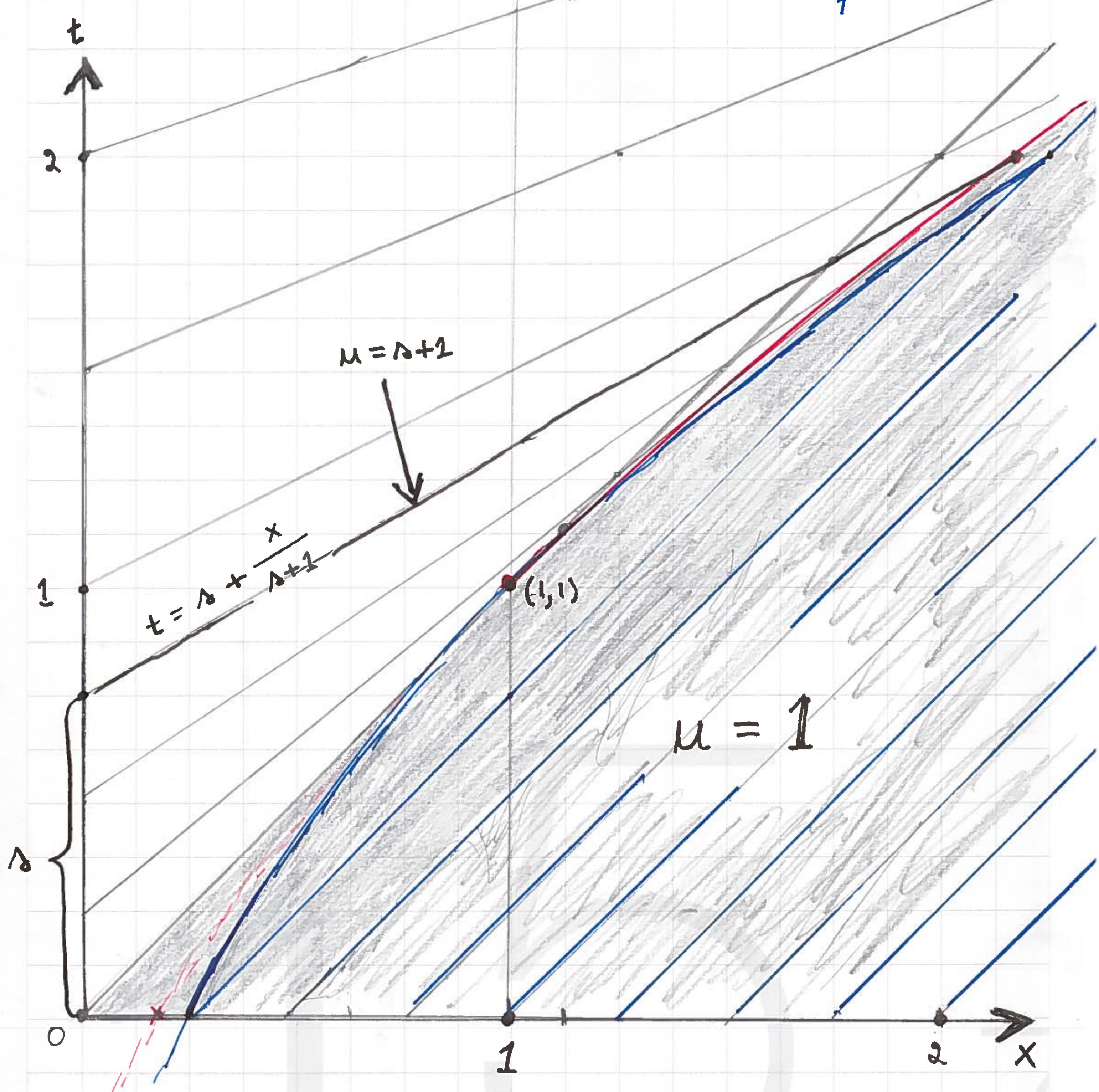
$$\boxed{\text{SHOCK} \qquad 16x = (3t+1)(t+3)}$$

The point $x=1$, $t=1$ is its endpoint, because the characteristics satisfy $t \geq x$ when $0 < x \leq 1$.

$$u(x,t) \overset{?}{=} \begin{cases} \dfrac{t+1}{2} + \sqrt{\left(\dfrac{t+1}{2}\right)^2 - x}\,; & 16x < (3t+1)(t+3) \\[2ex] 1 \quad \text{when} & 16x > (3t+1)(t+3) \end{cases}$$

DIAGRAM

Peter Lindqvist 2016



SHOCK $16x = (3t+1)(t+3)$, $x \geq 1$, $t \geq 1$

ENVELOPE: $x = \left(\frac{t+1}{2}\right)^2$ (below the shock)

of the characteristics

$t = s + \frac{x}{s+1}$ (they intersect here)