(1) Assume that $u_j \longrightarrow u$ weakly in $L^2(\Omega)$, i.e.,

$$\lim_{j\to\infty} \int_\Omega u_j v \, dx = \int_\Omega u v \, dx$$

for all $v \in L^2(\Omega)$. Show that, then,

$$\lim_{j\to\infty} \| u_j - u \|_{L^2(\Omega)} = 0 \iff \lim_{j\to\infty} \| u_j \|_{L^2(\Omega)} = \| u \|_{L^2(\Omega)}.$$

(2) Solve

$$\begin{cases} u_t + t u_x = u^2 \\ u(x,0) = \dfrac{1}{1+x^2} \end{cases}$$

(3) Find the general solution of Thomas' Eqn

$$u_{xt} = u_x u_t - 1$$

<u>Hint:</u> $v = f(u)$.

(4) Verify that $u(x,t) = -\dfrac{x}{\sqrt{16\pi t^3}} e^{-x^2/4t}$

solves the Heat Eqn $u_t = u_{xx}$ when $-\infty < x < \infty$, $t > 0$. Moreover,

$$\lim_{t\to 0+} u(x,t) \equiv 0.$$

(5) Problem:

$$\begin{cases} u_t + u u_x = 0 & \text{when } x > 0,\ t > 0 \\ u(x,0) = 1, & x > 0 \\ u(0,t) = 1 + t, & t > 0 \end{cases}$$

Sketch a "characteristic diagram". Find the breaking time t^*. Determine the solution up to t^*. After the shock forms, calculate its path in the xt-plane.

Answer: $16x = (t+3)(3t+1)$