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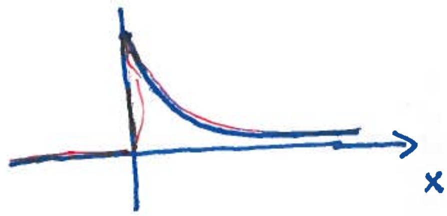
$$\boxed{\begin{array}{l} u_t = k u_{xx} \\ -\infty < x < \infty \end{array}}$$

Heaviside function.

(6) $$u(x,0) = H(x)\, e^{-x}$$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-y} e^{-\frac{(x-y)^2}{4kt}}\, dy$$



complete the square

$$y + \frac{(x-y)^2}{4kt} = \frac{4kty + x^2 - 2xy + y^2}{4kt}$$

$$= \frac{y^2 - 2y(x-2kt) + (x-2kt)^2}{4kt} + \underbrace{\frac{x^2 - (x-2kt)^2}{4kt}}_{(x-kt)}$$

$$= \frac{(y - x + 2kt)^2}{4kt} + (x - kt)$$

$$u(x,t) = e^{(kt-x)} \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(y-x+2kt)^2}{4kt}}\, dy$$

$$= e^{(kt-x)} \frac{1}{\sqrt{\pi}} \int_{-\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-z^2}\, dz$$

Boundary values:

$$-\frac{x-2kt}{\sqrt{4kt}} \longrightarrow \begin{cases} -\infty, & x > 0 \\ +\infty, & x < 0 \end{cases}$$

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(5) $$u(x,t) = e^{px} e^{kp^2 t}$$

As above in (6) with obvious changes!

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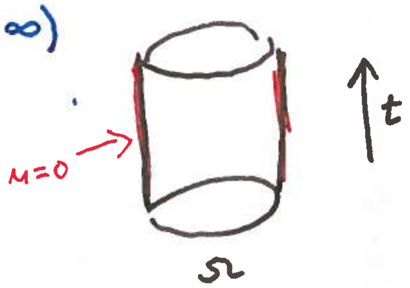
Diffusion coefficient.

(7) $$u_t = \nabla\cdot\left(k(\bar{x},u)\nabla u\right) \quad \text{in } \Omega\times(0,\infty)$$

$$u(\bar{x},t) = 0 \quad \text{when } \bar{x}\in\partial\Omega$$



$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega} u(\bar{x},t)^2\, d\bar{x}$$

$$\dot{\mathcal{E}}(t) = \frac{2}{2}\int_{\Omega} u u_t\, d\bar{x} = \frac{2}{2}\int_{\Omega} u\, \underline{\nabla\cdot\left(k(\bar{x},u)\nabla u\right)}\, d\bar{x}$$

Outer normal.

$$\overset{*)}{=} -\frac{1}{2}\int_{\Omega} k|\nabla u|^2\, d\bar{x} + \int_{\partial\Omega} \underline{u}\, k\, \nabla u\cdot\bar{n}\, dS$$

$u = 0$ on $\partial\Omega$

$$\leq 0.$$

$$\nabla\cdot\left[u k \nabla u\right] = \underline{u\,\nabla\cdot(k\nabla u)} + k\,\nabla u\cdot\nabla u$$

<u>Hence</u> $\mathcal{E}(t)$ is decreasing and

$$0 \leq \mathcal{E}(t) \leq \mathcal{E}(0+).$$

*) Divergence theorem (Gauß)

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(8) $u_{xxx} \equiv 0$

$$u(x,t) = c(t)x^2 + b(t)x + a(t)$$

$$= \underline{x^2 + 2kt}$$

(9) a) $$u(x,t) = e^{-dt} \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} g(y) e^{-\frac{(x-y)^2}{4kt}} dy$$

b) Influences the speed of decay.

c) $u_t + d(t)u = k u_{xx}$

Ansatz: $u(x,t) = \phi(t) v(x,t)$

$$\phi'(t) v + \phi(t) v_t = k \phi(t) v_{xx}(x,t)$$
$$+ d(t)\phi(t) v$$

$$\phi' + d(t)\phi = 0$$

$$\phi = C e^{-\int d(t) dt}$$

$$\phi(t) = e^{-\int_0^t d(\tau) d\tau}$$

$$\boxed{\begin{aligned} u(x,t) &= e^{-\int_0^t d(\tau) d\tau} v(x,t) \\ v_t &= k v_{xx} \\ u_t + d(t) u &= k u_{xx} \end{aligned}}$$