Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4305 Partial Differential Equations

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Examination date: 19 December 2016

Examination time (from–to): 09:00 – 13:00

Permitted examination support material: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he wants. No other aids permitted.

Other information:
There are 7 problems of equal weight: 1, 2, 3, 4, 5, 6a, 6b.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☐
sort/hvit ☐	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Solve the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + tx^2 \\ v(x,0) = 0 \end{cases}$$

where $-\infty < x < \infty,\ t > 0$.

Problem 2 Consider the equation

$$u_t = u_{xx} + 2u_{yy}$$

in the domain $\Omega_T \equiv \Omega \times (0,T)$, where Ω is a bounded domain in the $xy-$plane. Formulate and prove the *parabolic* maximum principle for the solutions. (You may assume[1] that $u \in C^2(\overline{\Omega_T})$.)

Problem 3 Show that the problem

$$\begin{cases} u_{tt} - c^2\,\Delta u + 3u = F(\mathbf{x},t), \\ u(\mathbf{x},0) = f(\mathbf{x}) \\ u_t(\mathbf{x},0) = g(\mathbf{x}) \end{cases}$$

can have at most one smooth solution $u = u(\mathbf{x},t)$ in $\mathbb{R}^3 \times (0,T)$. Assume that for each fixed $t \geq 0$ there is a radius R_t such that $u(\mathbf{x},t) = 0$ when $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \geq R_t$. *Hint:* Energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(w_t^2 + c^2|\nabla w|^2 + 3w^2\right) dx_1 dx_2 dx_3.$$

Problem 4 Determine the constants C and α so that $u(x) = Ce^{\alpha|x|}$ is a solution of the equation

$$u''(x) - k^2 u(x) = -\delta(x)$$

in the sense of distributions. Use test functions in $C_0^\infty(\mathbb{R})$.

[1] The notation means that u has continuous second derivatives up to the boundary

Problem 5 The function

$$u(x,t) = \begin{cases} \dfrac{x-2}{t+2}, & \text{when} \quad x > \xi(t) \\ 0, & \text{when} \quad x < \xi(t) \end{cases}$$

is a weak solution to Burgers's equation

$$u_t + uu_x = 0$$

in the upper half-plane $t > 0$. The unknown shock curve $x = \xi(t)$ starts at the origin. Determine the shock curve and draw a picture with characteristics in the xt-plane.

Problem 6 Assume that the variational integral

$$I(u) = \int_0^1 \int_0^1 \left(e^{2x} u_x^2 + u_x u_y + u_y^2 - 2u^3 \right) dx\, dy$$

attains its minimum among all smooth functions with boundary values 0 on the sides of the square.

a) Prove that the minimizer is unique.

b) Derive a second order equation for the minimizer (the Euler–Lagrange Equation). Then, prove that the minimizer is non-negative.

Good luck!