WAVE EQUATION. UNIQUENESS.

$$u_{tt} = c^2 \Delta u$$

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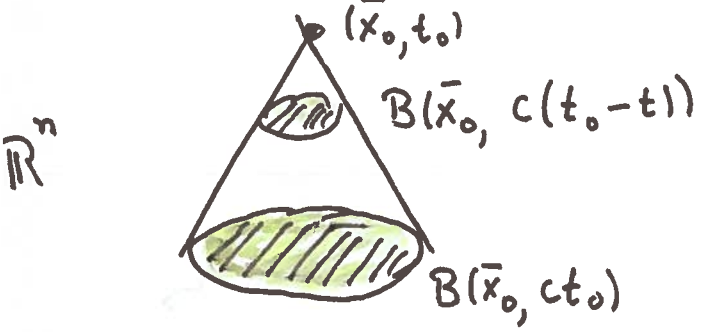
DOMAIN OF DEPENDENCE



The "light cone" $\overline{\Omega_{\bar{x}_0,t_0}}$ is defined via

$$|\bar{x} - \bar{x}_0| \le c(t - t_0),$$
$$0 \le t \le t_0$$

Define the "energy"

$$\mathcal{E}(t) = \frac{1}{2} \int_{B(\bar{x}_0, c(t-t_0))} \left(u_t^2(\bar{x}, t) + c^2 |\nabla u(\bar{x}, t)|^2 \right) d\bar{x}$$

for the solution $u \in C^2(\overline{\Omega_{\bar{x}_0,t_0}})$ of $u_{tt} = c^2 \Delta u$.

$$\dot{\mathcal{E}}(t) = \int_{B(\bar{x}_0, c(t-t_0))} \left(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \right) d\bar{x}$$

$$- \frac{c}{2} \oint_{S(\bar{x}_0, c(t-t_0))} \left(u_t^2 + c^2 |\nabla u|^2 \right) dS_{c(t-t_0)}$$

We do not need anything from the outside of the cone. (A local solution!)

We claim that $\dot{\mathcal{E}}(t) \le 0$. Now

$$\operatorname{div}(u_t \nabla u) = \nabla u_t \cdot \nabla u + u_t \Delta u$$

$$\int_{B(x_0, c(t-t_0))} \operatorname{div}(u_t \nabla u)\, d\bar{x} = \oint_{S(x_0, c(t-t_0))} u_t (\nabla u \cdot \bar{n})\, dS_{c(t-t_0)}$$

Divergence Theorem!

$|\bar{n}| = 1$

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Thus

$$\dot{\mathcal{E}}(t) = \int_{B(\bar{x}_0,\, c(t-t_0))} u_t \underbrace{(u_{tt} - c^2 \Delta u)}_{=0} \, d\bar{x}$$

$$+ \frac{c}{2} \underbrace{\int_{S(\bar{x}_0,\, c(t-t_0))} \left(2 u_t \frac{\partial u}{\partial \bar{n}} c - u_t^2 - c^2 |\nabla u|^2 \right) dS_{c(t-t_0)}}_{\leq 0 \text{ since}}$$

$$|2 u_t \nabla u \cdot \bar{n}\, c| \leq u_t^2 + c^2 |\nabla u \cdot \bar{n}|^2$$

$$\leq u_t^2 + c^2 |\nabla u|^2 \qquad 2ab \leq a^2 + b^2$$

$\underline{\dot{\mathcal{E}}(t) \leq 0}$.

Therefore $\quad \mathcal{E}(t) \leq \mathcal{E}(0)$

THM If $u(x,0) = 0$, $u_t(x,0) = 0$, when $|x - x_0| \leq c t_0$, then $u(x,t) = 0$ in the cone

$$|x - x_0| \leq c(t - t_0), \quad 0 \leq t \leq t_0.$$

In particular, $u(x_0, t_0) = 0$.

<u>Proof</u>: $\mathcal{E}(t) \leq \mathcal{E}(0) = 0$, since also $\nabla u(x,0) = 0$. Then $u_t^2 + c^2 |\nabla u|^2 = 0$ in the cone. Thus $u_t = 0$, $\nabla u = 0$ in the cone. Hence u is constant. The constant is zero. $\square$