Peter Lindqvist 1994

SHOCK FORMATION (FROM SMOOTH DATA), BREAKING TIME

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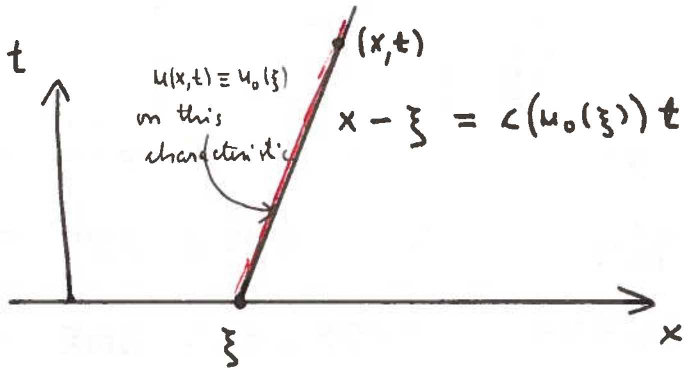
$$\begin{cases} u_t + c(u)\,u_x = 0 & (-\infty < x < \infty,\ t > 0) \\ u(x,0) = u_0(x) \end{cases}$$

Assume that $c(u) > 0$, $\underline{c'(u) > 0}$, and $u_0 \in C^1$.



$u = u_0(\xi)$ along $\underline{x - \xi = c(u_0(\xi))\, t}$

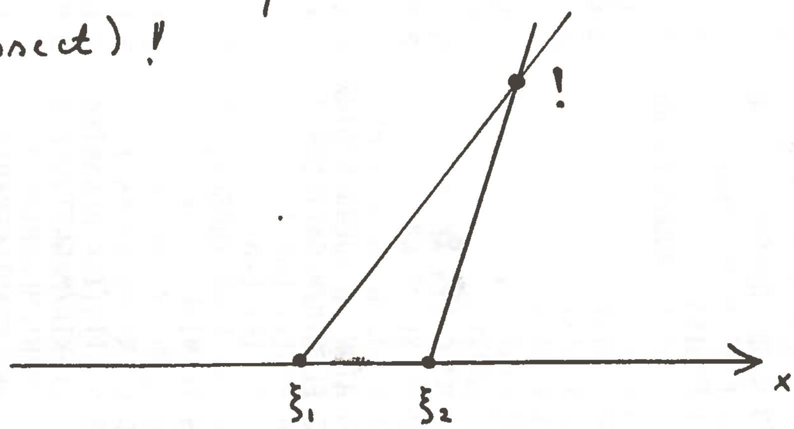
$u_0'(\xi) < 0$

$\xi_1 < \xi_2 \implies c(u_0(\xi_1)) > c(u_0(\xi_2))$, if u_0 is decreasing.

The characteristic emanating from ξ_1 is faster than the one emanating from ξ_2. (Remember that

$$s = \frac{dx}{dt} = c(u_0(\xi))$$

is the speed.) Therefore the characteristics will cross (intersect)!



To determine the breaking time, when a gradient catastrophe occurs, we calculate u_x along a characteristic

$$x - \xi = c(u_0(\xi))\, t.$$

For $g(t) = u_x(x(t), t)$ we have

$$g'(t) = u_{xx}\, x'(t) + u_{xt}$$

$x' = c(u_0(\xi))$

$$= \underline{u_{xx}\, c(u_0) + u_{xt}}.$$

Differentiating the Diff. Eqn. with respect to x yields

$$\underline{u_{tx} + c(u)\, u_{xx}} + c'(u)\, u_x^2 = 0.$$

Thus

$$g'(t) = -\overset{-c'(u_0(\xi))}{c'(u_0)}\, g^2(t)$$

with $u = u(x(t), t)$. Integrating this ordinary differential equation, we get

$$\int_{g(0)}^{g(t)} \frac{dg}{g^2} = -\int_0^t c'(u_0)\, dt,$$

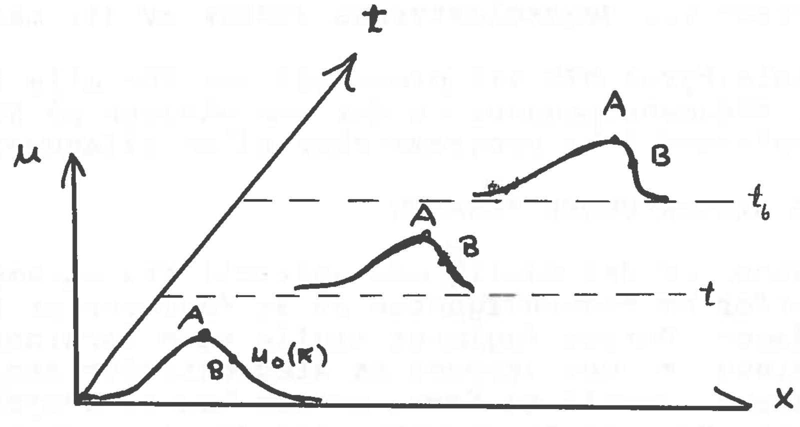
Here $c'(u)$ is a constant, viz. $c'(u_0(\xi))$.

$$g(t) = \frac{g(0)}{1 + g(0)\, c'(u_0(\xi))\, t}, \quad \text{i.e.,}$$

$$u_x = \frac{u_0'(\xi)}{1 + \underbrace{u_0'(\xi)\, c'(u_0(\xi))}_{\substack{\ominus \quad \oplus \\ <0 \\ \text{by assumption}}}\, t}$$

along the characteristic. Hence u_x does not exist along this characteristic, when

$$t \geq -\frac{1}{u_0'(\xi)\, c'(u_0(\xi))} \overset{\text{OBS!}}{=} \frac{-1}{\frac{d}{d\xi}\, c(u_0(\xi))}$$



A travels faster than B. The wave steepens and evolves into a shock wave. At time t_b (breaking time) the gradient catastrophe $|u_x| = \infty$ occures.

Examining u_x along <u>all</u> characteristics, we find the <u>breaking time</u>

$$t_b = \min_{\xi} \left[\frac{-1}{\frac{d}{d\xi} c(u_0(\xi))} \right].$$

The corresponding ξ is found by maximizing $|F'(\xi)|$, where $F(\xi) = c(u_0(\xi))$. (Hence $F''(\xi) = 0$, $t_b = -1/F'(\xi)$.)

EX. $\begin{cases} u_t + u u_x = 0 & (-\infty < x < \infty,\ t > 0) \\ u(x,0) = e^{-x^2}. \end{cases}$

Draw a picture, please.

$$F(\xi) = c(u_0(\xi)) = e^{-\xi^2}$$

$$F'(\xi) = -2\xi e^{-\xi^2}$$

$$F''(\xi) = -(4\xi^2 - 2) e^{-\xi^2} = 0 \iff \xi = \pm\sqrt{\tfrac{1}{2}}$$

$$t_b = \frac{1}{2\sqrt{\frac{1}{2}}\, e^{-1/2}} = \sqrt{\frac{e}{2}} \approx 1{,}16.$$