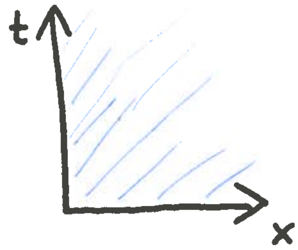
SOLUTION OF A DIFFERENTIAL EQN.

$$(*)\quad \begin{cases} \dfrac{\partial u}{\partial t} = x^2 \dfrac{\partial^2 u}{\partial x^2} + a x \dfrac{\partial u}{\partial x}, & \underline{t>0} \\ u(x,0) = f(x), & 0 < x < \infty \end{cases}$$



The transformation $\boxed{x = e^{-y}}$ $(-\infty < y < \infty)$

$$\mathcal{U}(y,t) = u(e^{-y}, t), \quad F(y) = f(e^{-y})$$

yields

$$\begin{cases} \dfrac{\partial \mathcal{U}}{\partial t} = \dfrac{\partial^2 \mathcal{U}}{\partial y^2} + (1-a)\dfrac{\partial \mathcal{U}}{\partial y}, & (t>0,\ -\infty<y<\infty) \\ \mathcal{U}(y,0) = F(y) \end{cases}$$

Use the Fourier transform

$$\hat{\mathcal{U}}(\xi,t) = \int_{-\infty}^{\infty} \mathcal{U}(y,t) e^{-2\pi i y \xi}\, d\xi. \qquad \hat{\mathcal{U}}(\xi,0) = \hat{F}(\xi)$$

(t = umbral variable)

Then

$$\widehat{\frac{\partial \mathcal{U}}{\partial t}} = \frac{\partial}{\partial t}\hat{\mathcal{U}}, \quad \widehat{\frac{\partial \mathcal{U}}{\partial y}} = 2\pi i \xi\, \hat{\mathcal{U}}, \quad \widehat{\frac{\partial^2 \mathcal{U}}{\partial y^2}} = -4\pi^2\xi^2\, \hat{\mathcal{U}}$$

and the equation becomes

$$\frac{\partial}{\partial t}\hat{\mathcal{U}} = [-4\pi^2\xi^2 + 2\pi i(1-a)\xi]\,\hat{\mathcal{U}},$$

$$\hat{\mathcal{U}}(\xi,t) = A(\xi)\, e^{[-4\pi^2\xi^2 + 2\pi i(1-a)\xi]t}$$

$$A(\xi) = \hat{\mathcal{U}}(\xi,0) = \hat{F}(\xi)$$

To invert

$$\hat{u}(\xi,t) = \hat{F}(\xi)\, e^{(-4\pi^2\xi^2 + 2\pi i (1-a)\xi)\,t} = \hat{F}(\xi)\hat{G}(\xi)$$

we need the inverse of the second factor above:

$$G(y) = \int_{-\infty}^{\infty} e^{-4\pi^2\xi^2 t + 2\pi i (1-a)\xi t}\, e^{+2\pi i \xi y}\, d\xi .$$

We find a diff. eqn for $G(y)$.

$$G'(y) = \int_{-\infty}^{\infty} \underbrace{2\pi i \xi\, e^{-4\pi^2\xi^2 t}}_{\frac{\partial}{\partial\xi}\left(-i\frac{e^{-4\pi^2\xi^2 t}}{4\pi t}\right)} e^{2\pi i (1-a)\xi t}\, e^{2\pi i \xi y}\, d\xi$$

(integration by parts)

$$= \underbrace{\left. -i\,\frac{e^{-4\pi^2\xi^2 t}}{4\pi t}\, e^{2\pi i (1-a)\xi t}\, e^{2\pi i \xi y}\right|_{-\infty}^{\infty}}_{=0\ \dagger)}$$

$$+ \frac{i}{4\pi t}\left[2\pi i (1-a)t + 2\pi i y\right] \underbrace{\int_{-\infty}^{\infty} e^{-4\pi^2\xi^2 t + 2\pi i (1-a)\xi t}\, e^{2\pi i \xi y}\, d\xi}_{G(y)}$$

†) The absolute value of the term in $\Big|_{-\infty}^{\infty}$ is $\frac{1}{4\pi t} e^{-4\pi^2\xi^2 t} \longrightarrow 0$ as $\xi \to \pm\infty$.

Thus

$$\boxed{G'(y) = \left(-\frac{y}{2t} - \frac{1-a}{2}\right) G(y)}$$

$$G(y) = C(t)\, e^{-\frac{y^2}{4t} - \frac{1-a}{2} y}$$

To find $C(t)$, set $y = 0$. Then

$$C(t) = G(0) = \int_{-\infty}^{\infty} e^{-4\pi^2\xi^2 t}\, e^{2\pi i (1-a)\xi t}\, d\xi .$$

Use $e^{-\pi \xi^2} = \widehat{e^{-\pi x^2}}$ to see that

$$e^{-\pi x^2} = \int_{-\infty}^{\infty} e^{-\pi \xi^2} e^{2\pi i \xi x}\, d\xi \qquad \xi = 2\sqrt{\pi t}\,\xi'$$

$$= 2\sqrt{\pi t} \int e^{-4\pi^2 \xi'^2 t} e^{2\pi i x \cdot 2\sqrt{\pi t}\,\xi'}\, d\xi'$$

Take $(1-a)t = 2x\sqrt{\pi t}$, $x = \frac{1-a}{2}\frac{\sqrt{t}}{\sqrt{\pi}}$. We get

$$C(t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(1-a)^2}{4}t}.$$

Recall that $\hat{U} = \hat{F}\hat{G}$

In toto $\overset{\text{CONVOLUTION}}{\boxed{U = F * G}}$, i.e.,

$$U(y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}(1-a)^2 t - \frac{y^2}{4t} - \frac{1-a}{2}y} * F(y)$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(y+(1-a)t)^2} * F(y)$$

$$= \int_{-\infty}^{\infty} G(y-z)F(z)\,dz$$

Subst $e^{-z} = v$

$-e^{-z}dz = dv$

$y - z = \ln v - \ln x$
$= \ln\left(\frac{v}{x}\right)$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{[\ln(\frac{v}{x}) + (1-a)t]^2}{4t}} \frac{f(v)}{v}\, dv \qquad x>0,\ t>0}$$

Black - Scholes' Eqn

$$\begin{cases} \dfrac{\partial V}{\partial t} + rs\dfrac{\partial V}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} - rV = 0 \\[2ex] V(s,T) = F(s)\,, \quad s > 0\,, \quad 0 < t < T \end{cases}$$

can be reduced to the previous situation.

1) $v(s,t) = V(s, T-t)\,, \qquad v(s,0) = F(s)$

$$\frac{\partial v}{\partial t} = rs\frac{\partial v}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} - rv$$

2) $u(s,t) = e^{+rt} v(s,t) \qquad u(s,0) = v(s,0) = F(s)$

This kills the rV term. Indeed,

$$\frac{\partial u}{\partial t} = rs\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2}$$

3) $w(s,t) = u\left(s, \frac{2t}{\sigma^2}\right) \qquad w(s,0) = F(s)$

$$\frac{\partial w}{\partial t} = s^2 w_{ss} + \frac{2r}{\sigma^2} s\, w_s$$

This we recognize as the eqn (*) with variables t, $x = s$ and parameter $a = \frac{2r}{\sigma^2}$.

Therefore

$$w(s,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \frac{F(v)}{v} e^{-\frac{\left[\ln\frac{v}{s} + \left(1 - \frac{2r}{\sigma^2}\right)t\right]^2}{4t}}\, dv$$

Now we redo the reductions:

$$u(s,t) = w\left(s, \frac{\sigma^2}{2} t\right)$$

$$v(s,t) = e^{-rt} u(s,t) = e^{-rt} w\left(s, \frac{\sigma^2 t}{2}\right)$$

$$V(s,t) = v(s, T-t)$$

$$= e^{-r(T-t)} w\left(s, \frac{\sigma^2}{2}(T-t)\right)$$

It follows that

$$\boxed{V(s,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_0^\infty e^{-\frac{\left[\ln\left(\frac{v}{s}\right) + \left(\frac{\sigma^2}{2} - r\right)(T-t)\right]^2}{2\sigma^2(T-t)}} F(v) \frac{dv}{v}}$$

This is BLACK-SCHOLES'S formula.

[? not in Stein]