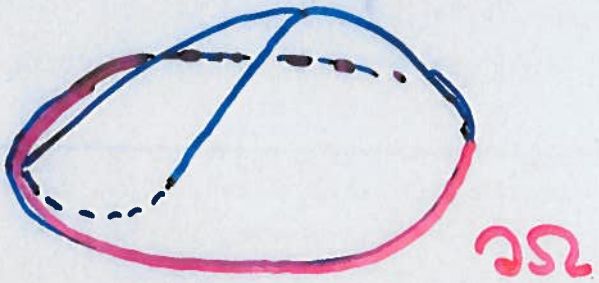
VIBRATING MEMBRANES

Ω = a bounded domain in the plane

$\partial\Omega$ = the boundary of Ω



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \quad ; \quad u(x,y,t) = 0 \text{ when } (x,y) \in \partial\Omega.$$

Separation $u(x,y,t) = v(x,y) \cdot T(t)$

$$vT'' = c^2 T \Delta v$$

$$\frac{\Delta v}{v} = \frac{T''}{c^2 T} = -\lambda \quad (\text{constant of separation})$$

$$\Delta v + \lambda v = 0, \quad v = 0 \text{ on } \partial\Omega$$

$$T'' + \lambda c^2 T = 0$$

If we multiply by v and integrate the resulting eqn $v\Delta v + \lambda v^2 = 0$ over Ω, we obtain

$$\lambda \iint_\Omega v^2 dx dy = -\iint_\Omega v \Delta v \, dx dy = \iint_\Omega |\nabla v|^2 dx dy$$

GREEN

so that

$$\lambda = \frac{\iint_{\Omega} |\nabla v|^2 \, dx\,dy}{\iint_{\Omega} v^2 \, dx\,dy} > 0$$

RAYLEIGH QUOTIENT

Here $|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$. Now

$$T'' + \lambda c^2 T = 0 \iff T = A\cos(c\sqrt{\lambda}\, t) + B\sin(c\sqrt{\lambda}\, t).$$

THEOREM The equation $\Delta v + \lambda v = 0$ has non-trivial solutions with zero boundary values only for certain values of λ, viz.

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \quad \text{EIGENVALUES}$$

where $\lambda_1 > 0$ (the principal frequency, the ground tone) and $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$. The corresponding solutions v_n are called EIGENFUNCTIONS; hence

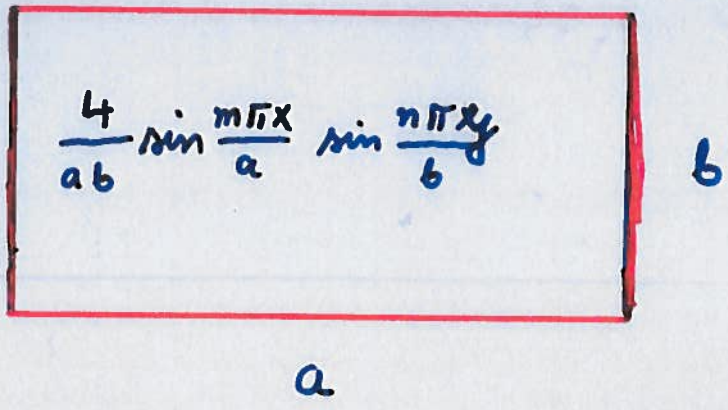
$$\Delta v_n + \lambda_n v_n = 0.$$

The general solution of the wave eqn is then

$$u(x,y,t) = \sum_{n=1}^{\infty} \left(a_n \cos(c\sqrt{\lambda_n}\, t) + b_n \sin(c\sqrt{\lambda_n}\, t)\right) u_n(x,y)$$

OBS! $u_1(x,y) > 0$ in Ω. All the other

EXAMPLE RECTANGULAR DRUM



$$u(x,y,t) = \sum_{m,n=1}^{\infty} \left[A_{mn}\cos(c\mu_{mn}t) + B_{mn}\sin(c\mu_{mn}t)\right] \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

$$\mu_{mn} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad (m,n = 1, 2, 3, \dots)$$

In the case

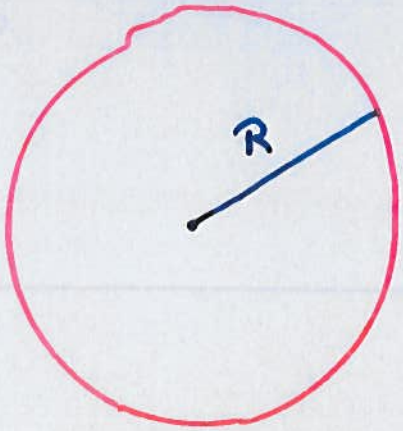
$$u(x,y,0) = f(x,y), \quad \frac{\partial u(x,y,0)}{\partial t} = 0$$

we have $B_{mn} = 0$ and

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

This comes from a "double" Fourier series.

CIRCULAR DRUM



$$\sqrt{\lambda_1} = 2,4048\ldots$$

$$\frac{\lambda_2}{\lambda_1} \approx \underbrace{2,5387\ldots}_{\S??}$$

This ratio for the circular drum is larger than for any other drum!

PAYNE-POLYA-WEINBERGER conjecture. Proved 1992

FABER-KRAHN INEQUALITY Among all domains with the same area the circle has the lowest ground tone.

$$\lambda_1(\Omega) \geq \lambda_1(\text{circle})$$

$$\lambda_1(\Omega) = \underset{\substack{v=0 \text{ on} \\ \partial\Omega}}{\text{MINIMUM}} \frac{\iint_\Omega |\nabla v|^2 \, dx\, dy}{\iint_\Omega v^2 \, dx\, dy}$$

H. WEYL 1911

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = \frac{4\pi}{\text{area}(\Omega)}$$

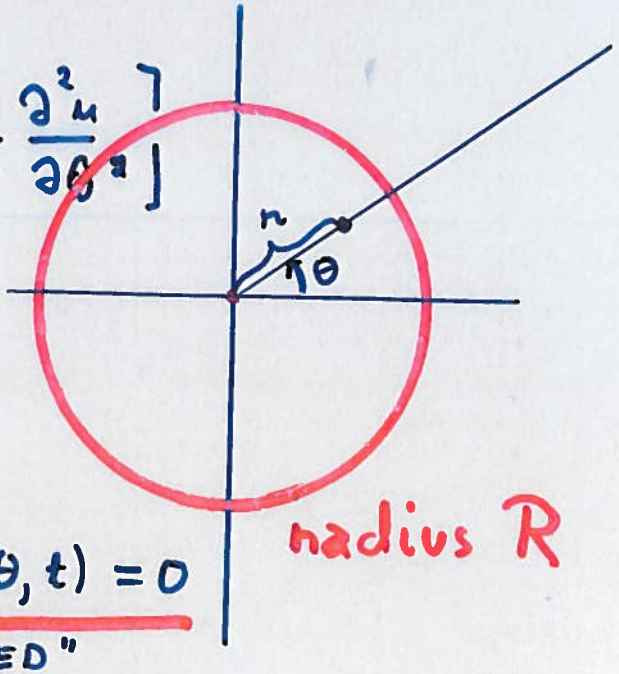
Valid for arbitrary domains!

One can hear the area of a domain!

THE VIBRATING CIRCULAR MEMBRANE

§11.10

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$



$u = u(r, \theta, t)$

INITIAL CONDITIONS

$$\begin{cases} u(r,\theta,0) = f(r,\theta) \\ u_t(r,\theta,0) = g(r,\theta) \end{cases}$$

$u(R,\theta,t) = 0$ "FIXED"

SIMPLIFICATION: Let us study only the radial vibrations, i.e., $u = u(r,t)$ and

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right].$$

This is a serious restriction, not physically justified.

SEPARATION OF VARIABLES

$$u(r,t) = W(r)\,T(t)$$

$$WT'' = c^2\left(W''T + \frac{1}{r}W'T\right)$$

$$\frac{T''}{c^2 T} = \frac{W''}{W} + \frac{W'}{rW} = -k^2$$

Constant of separation. It will be $k^2 > 0$

$$T'' + c^2 k^2 T = 0$$

$$W'' + \frac{W'}{r} + k^2 W = 0$$

SUBST. $s = kr$ $\quad W(s) = W(kr)$

$$\frac{dW}{ds} = \frac{dW}{dk} \cdot \frac{1}{k} \quad \text{etc}$$

$$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0, \quad W = W(s)$$

BESSEL'S EQUATION

The only bounded solutions are

$$\text{Const.} \cdot J_0(s)$$

Returning to the original variable r we have

$$W(r)\,T(t) = J_0(rk)\,T(t)$$

To satisfy the condition

$$J_0(Rk)\,T(t) \equiv 0$$

we get $Rk = \alpha_m$ (a zero of $J_0(r)$)

Hence, only the values

$$k = k_m = \frac{\alpha_m}{R} \quad (m = 1, 2, 3, \dots)$$

are possible.

We get the solutions

$$J_0\left(\frac{\alpha_m r}{R}\right)\cdot\left[A_m\cos\left(\frac{c\alpha_m}{R}t\right)+B_m\sin\left(\frac{c\alpha_m}{R}t\right)\right]$$

where $m = 1, 2, 3, \ldots$ A SUPERPOSITION of these yields all RADIAL SOLUTIONS

$$u(r,t) = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m r}{R}\right)\left[A_m\cos\left(\frac{c\alpha_m}{R}t\right)+B_m\cdots\right]$$

Unfortunately, the majority of the solutions are not of this form, except for radial initial conditions.

$t=0$ $\quad (0\le r<R)$

$$f(r) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m r}{R}\right)$$

$$g(r) = \sum_{m=1}^{\infty} B_m\cdot\frac{c\alpha_m}{R} J_0\left(\frac{\alpha_m r}{R}\right)$$

FOURIER-BESSEL SERIES

$$A_m = \frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R r f(r) J_0\left(\frac{\alpha_m}{R}r\right)dr$$

$$B_m\frac{c\alpha_m}{R} = \frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R r g(r) J_0\left(\frac{\alpha_m r}{R}\right)dr$$

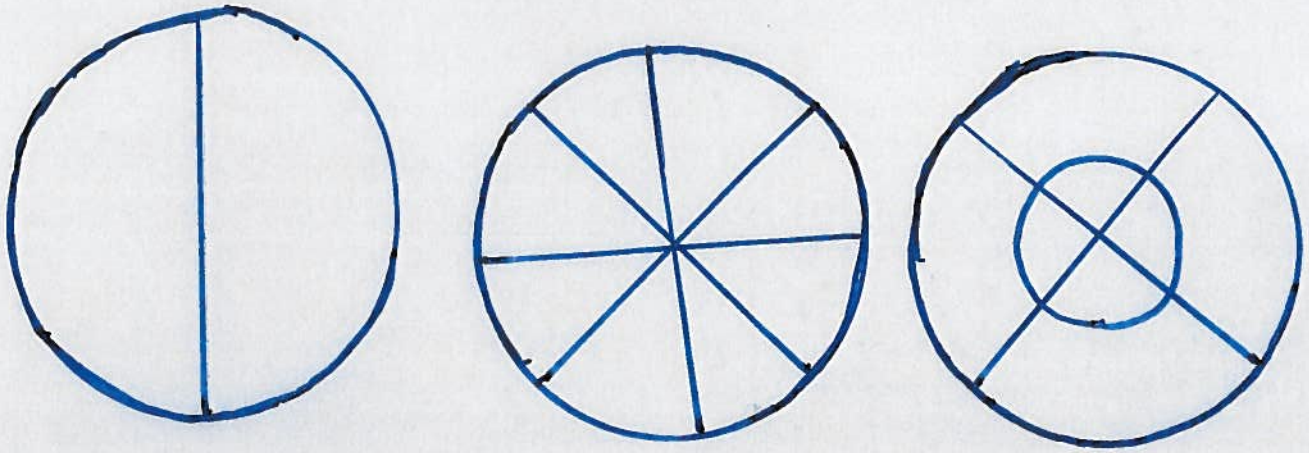
GENERAL CASE The Bessel functions $J_n(r)$ and their zeros

$$\alpha_{1n},\ \alpha_{2n},\ \alpha_{3n},\ \ldots,\ \alpha_{mn},\ \ldots$$

are needed. The general solution is a superposition of the solutions

$$\left[A_{mn}\cos\left(\frac{c\alpha_{mn}}{R}t\right) + B_{mn}\sin\left(\frac{c\alpha_{mn}}{R}t\right)\right] J_n\left(\frac{\alpha_{mn}r}{R}\right)\cos n\theta$$

$$\text{—— '' ——} \qquad \sin n\theta$$



We note that $J_n(x)$ is an even function of x when n is
:n, odd when n is odd.

The graphs of $J_0(x)$, $J_1(x)$ are indicated in Fig. 1.

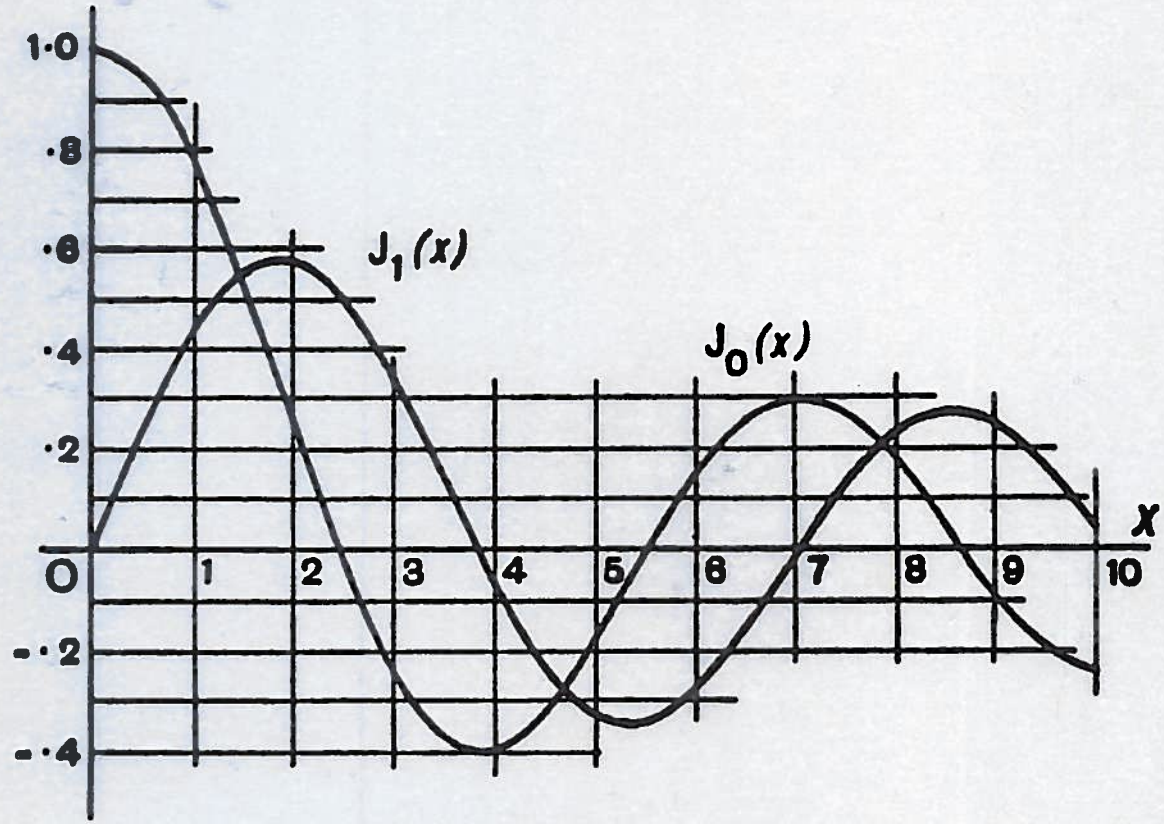


FIG. 1.

Extensive tables of values of $J_n(x)$, especially of $J_0(x)$
d $J_1(x)$, have been calculated on account of their applica-
)ns to physical problems.*

§ 3. *Bessel's equation of zero order.*

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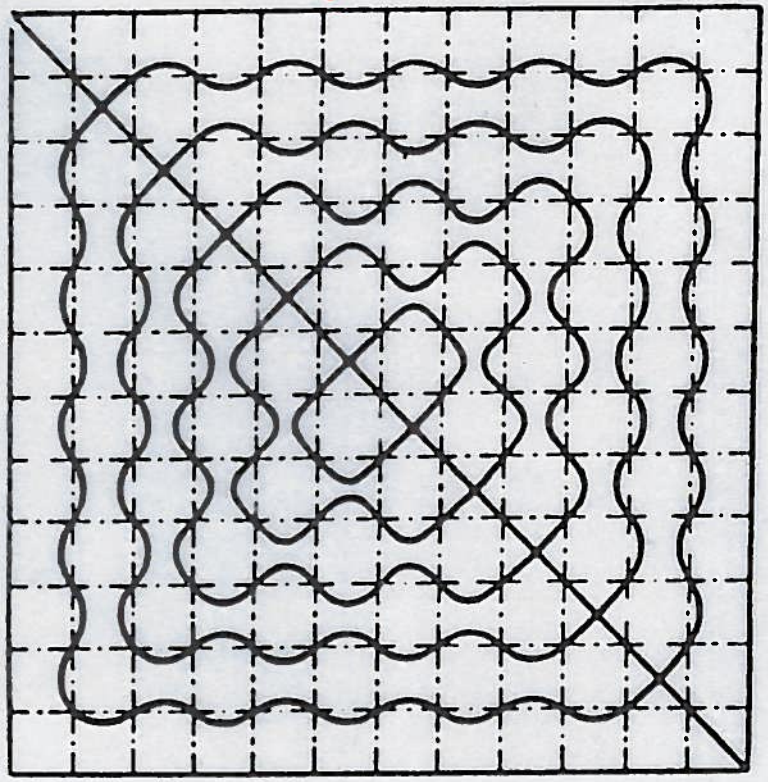
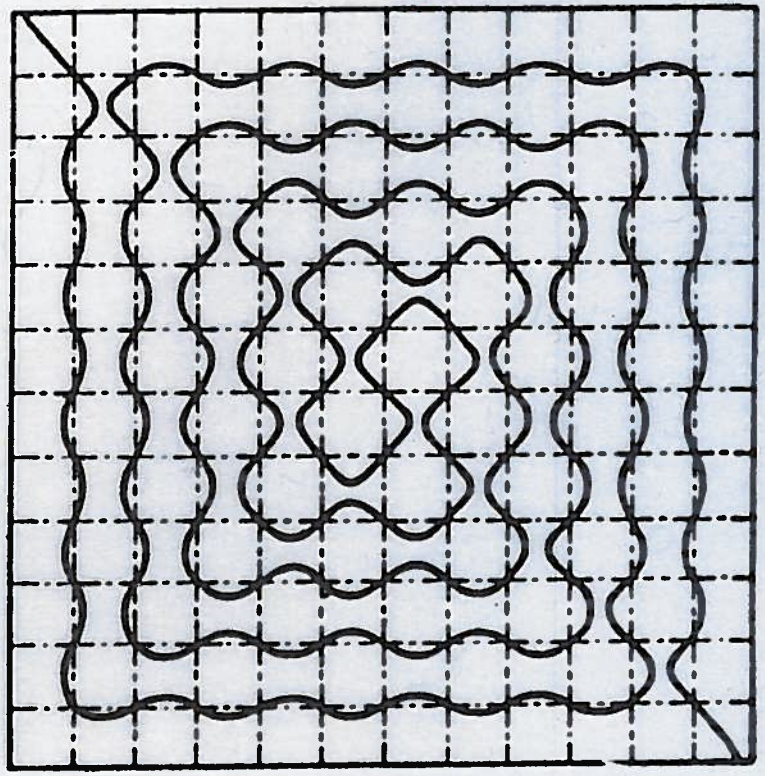
$\lambda = 24^2 + 1^2 = 577.$

12 subregions



2 subregions



$\sin 24x \sin y + \sin x \sin 24y$

Figure 2

$\sin 24x \sin y + b \cdot \sin x \sin 24y$

$b \approx 1$