

Ex. 3.3 $\Omega \subset \mathbb{R}^2$ a bounded domain and

$$\begin{cases} v_{xx} + v_{yy} = -2 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad v \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

Claim: $w = |\nabla v|^2$ attains its maximum on the boundary.

Proof: 1) $u = v + \frac{1}{2}(x^2 + y^2)$ is harmonic: $\Delta u = 0$.

2) The derivatives of harmonic functions are harmonic.

Thus $u_x = v_x + x$, $u_y = v_y + y$ are harmonic.

Then also v_x, v_y are harmonic functions ($v_x = u_x - x$ = a difference of harmonic functions)

3) $\Delta f = 0 \Rightarrow \Delta(f^2) = 2|\nabla f|^2 + 2f \Delta f = 2|\nabla f|^2 \geq 0$. Thus $\Delta(v_x^2) \geq 0$, $\Delta(v_y^2) \geq 0$.

Therefore

$$\Delta w = \Delta(v_x^2 + v_y^2) = \Delta(v_x^2) + \Delta(v_y^2) \geq 0 + 0 = 0$$

i.e. $w = |\nabla v|^2$ is subharmonic.

4) The maximum principle holds for subharmonic functions. Hence

$$w = |\nabla v|^2$$

attains its maximum on the boundary $\partial\Omega$.

Remark $v > 0$ in Ω . Why?