12.6/1d

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$$\begin{cases} u_{tt} = c^2 \Delta u \\ u = 0 \quad \text{when } t = 0 \\ u_t = \dfrac{1}{1+r^2}, \quad r^2 = x^2+y^2+z^2, \quad \text{when } t = 0 \end{cases}$$

(12.139) $$u = \frac{F(r-ct) + G(r+ct)}{r}$$

$$\begin{cases} F(r) + G(r) = 0 \\ \dfrac{c}{r}\left[-F'(r) + G'(r)\right] = \dfrac{1}{1+r^2} \end{cases}$$

$$G'(r) - F'(r) = \frac{r}{c(1+r^2)}$$

$$G(r) - F(r) = \frac{1}{2c}\ln(1+r^2) + \text{Const.}$$

$$G(r) = \frac{1}{4c}\ln(1+r^2) = -F(r)$$

Answer:

$$u = \frac{\ln[1+(r+ct)^2] - \ln[1+(r-ct)^2]}{4cr}$$

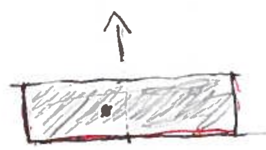
12.6/1a

$u(x,y,z,t) = x + z$

!

8.3/6 The maximum principle is violated for the solution $u(x,t) = -(x^2+2xt)$ in the domain

$$|x| \le 2, \quad 0 \le t \le 1.$$



Use $u(-\frac{1}{2}, \frac{1}{2})$! (The parabolic boundary values are ≤ 0.)

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8.3/4 $\boxed{u_t = u_{xx} \pm u_x}$

$$\Omega_T = (a,b) \times (0,T)$$

$$\overline{\Omega}_T = [a,b] \times [0,T]$$

Parabolic boundary: $\Gamma_T = ([a,b] \times \{0\}) \bigcup (\{a\} \times [0,T]) \bigcup (\{b\} \times [0,T])$

Maximum Principle $u \in C^2(\Omega_T)$, $u \in C(\overline{\Omega}_T)$, $u_t = u_{xx} + u_x$ in Ω_T. Then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

Proof: $\varepsilon > 0$. Define $v_\varepsilon = u - \dfrac{\varepsilon}{T-t}$.

$$(*) \quad \frac{\partial v_\varepsilon}{\partial t} = v_{\varepsilon,xx} + v_{\varepsilon,x} - \frac{\varepsilon}{(T-t)^2}$$

$$\leq v_{\varepsilon,xx} + v_{\varepsilon,x} - \frac{\varepsilon}{T^2} < v_{\varepsilon,xx} \pm v_{\varepsilon,x}$$

At an interior maximum of v_ε we would have

$$v_{\varepsilon,t} = 0, \quad v_{\varepsilon,x} = 0, \quad v_{\varepsilon,xx} \leq 0,$$

which contradicts $(*)$. Thus v_ε attains its maximum on the parabolic boundary Γ_T. The latest points $t = T$ are out of the question.

$$u(x,t) - \frac{\varepsilon}{T-t} = v_\varepsilon(x,t) \leq \max_{\Gamma_T} v_\varepsilon(x,t)$$

$$\leq \max_{\Gamma_T} u(x,t).$$

Let $\varepsilon \to 0$. Thus $u(x,t) \leq \max_{\Gamma_T} u$.

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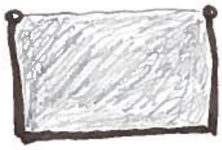
8.3/5

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \qquad 1 < x < 2, \quad 0 < t < T$$



The parabolic boundary Γ_T consists of the three segments

$$\begin{cases} 1 \leq x \leq 2, & t = 0 \\ x = 1, & 0 \leq t \leq T \\ x = 2, & 0 \leq t \leq T \end{cases}$$

<u>Claim</u> $\max\limits_{\substack{1 \leq x \leq 2 \\ 0 \leq t \leq T}}\{u(x,t)\} = \max\limits_{\Gamma_T} u \qquad u \in C^2(\Omega_T)$, $u \in C(\overline{\Omega_T})$

<u>Proof:</u> $v^\varepsilon(x,t) = u(x,t) - \dfrac{\varepsilon}{T-t}$

$$\frac{\partial v^\varepsilon}{\partial t} = x\, v^\varepsilon(x,t)_{xx} + v^\varepsilon(x,t)_x - \frac{\varepsilon}{(T-t)^2}$$

$$\leq \text{———} \;\; '' \;\; \text{———} - \varepsilon/T^2$$

$$< x\, v^\varepsilon(x,t)_{xx} + v^\varepsilon(x,t)_x$$

At an interior maximum of v^ε we would have

$$v^\varepsilon_t = 0, \quad v^\varepsilon_x = 0, \quad v^\varepsilon_{xx} \leq 0$$

and hence

$$0 < x\, v^\varepsilon(x,t)_{xx} + 0 \leq 0 \quad (\text{since } x > 0),$$

a contradiction. Thus the maximum of v^ε is on the boundary. It must be on the <u>parabolic</u> boundary.

Let $\varepsilon \to 0$ in

$$u(x,t) - \frac{\varepsilon}{T-t} \leq \max_{\Gamma_T} v^\varepsilon \leq \max_{\Gamma_T} u .$$

The rest is clear.